

## SDiff gauge theory and the M2 condensate

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## SDiff gauge theory and the M2 condensate

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**ABSTRACT:** We develop a general formalism for the construction, in  $D$ -dimensional Minkowski space, of gauge theories for which the gauge group is the infinite-dimensional group  $\text{SDiff}_n$  of volume-preserving diffeomorphisms of some closed  $n$ -dimensional manifold. We then focus on the  $D = 3$   $\text{SDiff}_3$  superconformal gauge theory describing a condensate of M2-branes; in particular, we derive its  $\mathcal{N} = 8$  superfield equations from a pure-spinor superspace action, and we describe its relationship to the  $D = 3$   $\text{SDiff}_2$  super-Yang-Mills theory describing a condensate of D2-branes.

**KEYWORDS:** Gauge Symmetry, Superspaces, Conformal Field Models in String Theory, M-Theory.

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## 1. Introduction

The M2-branes of M-theory may have boundaries on an M5-brane because the M2-charge can be taken up by the 2-form gauge potential on the M5-brane worldvolume [1, 2]. Following the determination of the M5-brane equations of motion [3] and the construction of its action [4], it was verified that there exists a ‘soliton-type’ solution with this interpretation [5]. This possibility can also be understood from the M2-brane perspective in

terms of its superalgebra [6], and is realizable in terms of an open membrane subject to appropriate boundary conditions [7] but not, for a single M2-brane, as a ‘soliton-type’ solution of the M2-brane equations of motion. This is hardly surprising given the disparity in dimension but one may imagine that multiple M2-branes could expand to generate the required extra dimensions as a ‘fuzzy’ 3-sphere, and an equation that might describe such a configuration was proposed by Basu and Harvey [8]. This equation led Bagger and Lambert to propose [9], as a low-energy limit of coincident planar M2-branes, a novel class of 3-dimensional maximally supersymmetric gauge theories based on Filippov 3-algebras, rather than Lie algebras; a similar framework was developed by Gustavsson [10]. Such gauge theories have the  $OSp(8|4)$  superconformal symmetry expected of an action for multiple M2-branes in a low-energy limit [11], and they admit the Basu-Harvey equation as a ‘BPS’ equation.

Explicit realizations of the Bagger-Lambert-Gustavsson (BLG) theory arise from specific Filippov 3-algebras. A particular 4-dimensional example,  $\mathcal{A}_4$ , was considered by Bagger and Lambert [9] but the corresponding BLG model has since been shown [12, 13] to describe the dynamics of two M2-branes on an orbifold rather than flat space. This model is also disappointing in one other respect: it is equivalent to a ‘standard’ Chern-Simons (CS) theory for gauge group  $SU(2) \times SU(2)$  coupled to  $\mathcal{N} = 8$  matter multiplets in the  $(\mathbf{2}, \mathbf{2})$  representation [14], so the novel algebraic structure of the general construction plays no essential role in this example. Furthermore, all other finite-dimensional Filippov ‘metric’ 3-algebras (those with positive definite algebra-compatible metric) are direct sums of  $\mathcal{A}_4$  and trivial one-dimensional 3-algebras [15, 16], so the nature of the action describing the low-energy dynamics of an arbitrary finite number  $N$  of coincident planar M2-branes remains an unsolved problem, although there is no shortage of proposals. We will return to this point at the conclusion of this paper; for most purposes here it is sufficient that there are clear candidates for the  $N \rightarrow \infty$  limit, which one can view as describing possible ‘condensates’ of coincident planar M2-branes. These are the BLG theories in which the Filippov 3-algebra is realized by the Nambu-bracket [17] of functions defined on some 3-manifold  $M_3$ ; the choice  $M_3 = S^3$  then leads to a version of the Basu-Harvey equation in which the fuzzy 3-sphere becomes a classical 3-sphere [18].

Recall that the Nambu  $n$ -bracket for  $n$  functions  $(\phi^1, \dots, \phi^n)$  on a closed  $n$ -dimensional manifold  $M_n$  with coordinates  $\sigma^i$  ( $i = 1, \dots, n$ ) is

$$\{\phi^1, \dots, \phi^n\} = e^{-1} \varepsilon^{i_1 \dots i_n} \partial_{i_1} \phi^1 \dots \partial_{i_n} \phi^n, \tag{1.1}$$

where  $\varepsilon$  is the invariant antisymmetric tensor density on  $M_n$ . We choose to define this bracket as a scalar on  $M_n$  by dividing by some fixed scalar density  $e$  on  $M_n$ . The space of functions on  $M_n$  can then be viewed as an infinite-dimensional ‘ $n$ -algebra’. This algebra obeys a ‘fundamental’ identity that can be expressed simply in terms of two *anticommuting* (‘ghost’) functions  $(B, C)$  on  $M_n$ :

$$\{B, \dots, B, \{C, \dots, C\}\} = n \{ \{B, \dots, B, C\}, C, \dots, C \}. \tag{1.2}$$

Abstractly, any  $n$ -algebra defined by an  $n$ -linear antisymmetric product that obeys the fundamental identity is a Filippov  $n$ -algebra. The  $n$ -algebra of the space of func-

tions on  $M_n$  with respect to the Nambu  $n$ -bracket is therefore an infinite-dimensional Filippov  $n$ -algebra.

One should not think of the density  $e$  on  $M_n$  as derived from a metric on  $M_n$  because no metric will be used in our constructions, but one may choose  $e$  to coincide with  $\sqrt{g}$  for some ‘fiducial’ metric  $g$  that one could introduce for this purpose. For example, if  $M_n \cong S^n$  then one may choose  $e = \sqrt{g}$  where  $g$  is the  $SO(n+1)$ -invariant metric on the unit  $n$ -sphere. This choice facilitates the identification of the finite-dimensional sub-algebra that exists when  $M_n \cong S^n$ . Consider  $(n+1)$  functions  $X_a$  ( $a = 1, \dots, n+1$ ) subject to the constraint

$$\sum_{a=1}^{n+1} X_a^2 = 1. \tag{1.3}$$

Given that  $e$  has been chosen as specified above, then

$$\{X_{a_1}, \dots, X_{a_n}\} = \epsilon^{a_1 \dots a_n a_{n+1}} X_{a_{n+1}}, \tag{1.4}$$

which shows that the  $X_a$  span an  $(n+1)$ -dimensional subalgebra:  $\mathcal{A}_n$ . For  $n = 2$  the Nambu bracket is a Poisson bracket and we therefore have a realization of the Lie algebra  $su(2)$  by functions on  $S^2$ , so  $\mathcal{A}_3 = su(2)$ . For  $n = 3$  we have a realization of the four-dimensional Filippov 3-algebra  $\mathcal{A}_4$  by functions on  $S^3$ .

As suggested in [18] and shown in [19–21], the Nambu bracket realization of the BLG theory is an ‘exotic’ gauge theory for the group  $\text{SDiff}(S^3)$  of volume-preserving diffeomorphisms of the 3-sphere. A rather explicit discussion of this group is given in [22]; other 3-manifolds  $M_3$  yield slightly different theories; we return to this point in the final section but otherwise pass over it, using the notation  $\text{SDiff}_3$  for the group of volume-preserving diffeomorphisms of any closed 3-manifold  $M_3$ . We say that  $\text{SDiff}_3$  gauge theories are ‘exotic’ because they cannot be obtained from an ‘abstract’ YM theory, whereas this is possible for  $\text{SDiff}_2$  gauge theories; we elaborate on this this point later. Since the fields of an  $\text{SDiff}_3$  gauge theory also depend on the three coordinates of  $M_3$ , the Nambu bracket realization of the BLG theory is effectively a 6-dimensional theory. It has been suggested that this is a version of the M5-brane action [19, 20], although the most straightforward way to extract an  $\text{SDiff}_3$  gauge theory from the *standard* M5-brane action leads to the Carrollian limit of the BLG theory [21].

The main aim of this paper is to put the Nambu-bracket realization of the BLG theory into a larger context by developing further the general principles of  $\text{SDiff}$  gauge theory. It is well-known that  $\text{SDiff}_2$  gauge theories may loosely be considered as  $N \rightarrow \infty$  limits of  $SU(N)$  gauge theories in which the matrix commutator becomes the Poisson bracket of functions on a 2-manifold [23], the 2-sphere being the simplest case. Such theories first arose from light-cone gauge-fixing of a relativistic membrane, and the application to the M2-brane yields a maximally supersymmetric gauge mechanics model in which the gauge group is the infinite-dimensional group of area-preserving diffeomorphisms of the membrane. In the case of a spherical membrane, there is a sequence of truncations of the group of area-preserving diffeomorphisms to  $SU(N)$  that reduces the membrane action to the action for a maximally-supersymmetric  $SU(N)$  gauge mechanics model [24]; this truncation is one in

which the classical 2-sphere is replaced by a fuzzy sphere [25]. The truncated model can be interpreted as describing the dynamics of multiple D0-branes [2], and is the basis of the M(matrix) model formulation of M-theory [26].

In the context of gauge mechanics models, which we may view as examples of  $D$ -dimensional gauge theories for  $D = 1$ , there exist  $\text{SDiff}_n$  gauge theories for any  $n = p$  obtained by the light-cone gauge-fixing of the action for a relativistic  $p$ -brane [27] (although supersymmetry constrains  $p$  and hence  $n$ ). What we are interested in this paper is how  $\text{SDiff}_n$  gauge theories may be constructed for  $D > 1$ . The answer to this question for  $n = 2$  is known. Because  $\text{SDiff}_2$  gauge theories are just standard, albeit infinite-dimensional, Yang-Mills theories, any Yang-Mills theory that can be constructed for all  $\text{SU}(N)$  can also be constructed for  $\text{SDiff}_2$  [28]. For example, one may choose the gauge group for the  $D = 4$   $\mathcal{N} = 4$  super-Yang-Mills theory to be  $\text{SDiff}(S^2)$ , in which case we have a 6-dimensional theory. It is possible that this is related to the M5-brane in the much the same way as the Nambu-bracket realization of the BLG theory, but we shall not investigate this possibility here. Instead, we focus on possibilities for  $\text{SDiff}_n$  gauge theories with  $n > 2$ .

It appears that there are no useful possibilities for  $n \geq 4$  because of the difficulty in constructing a kinetic term for the gauge potential without a metric on  $M_n$ . For this reason, we focus on the  $n = 3$  case. Remarkably,  $\text{SDiff}_3$  gauge theories may be constructed for any spacetime dimension  $D$  in close analogy to Yang-Mills theory, although these theories are still ‘exotic’ in the sense explained above. However, they are unlikely to be of any physical relevance because their energy density is not positive definite. For  $D = 3$  there is another option: one may construct a Chern-Simons-type term. This leads to a new class of (super)conformal  $D = 3$  gauge theories, which we focus on in this paper. The Nambu bracket realization of the BLG theory is the maximally-supersymmetric  $\text{SDiff}_3$  gauge theory of this type, and we re-construct it from our formalism, presenting simple proofs of both its  $\mathcal{N} = 8$  supersymmetry and its superconformal invariance. Although there is no free field limit of the BLG action, we show that one can take a free-field limit of the equations of motion, in which case one arrives at a theory for an infinite number of non-interacting  $\mathcal{N} = 8$  scalar supermultiplets related by a *rigid*  $\text{SDiff}_3$  symmetry.

As we are attempting to put the BLG model into a more general context, we consider the general construction of superconformal  $\text{SDiff}_3$  gauge theories in terms of  $\mathcal{N} = 1$  superfields.<sup>1</sup> Obviously, any (Minkowski space)  $\text{SDiff}_3$  gauge theory with  $\mathcal{N} > 1$  supersymmetry can be written in terms of  $\mathcal{N} = 1$  superfields, although the extended supersymmetry will not then be manifest. To make the  $\mathcal{N} = 8$  supersymmetry of the BLG theory manifest, one needs a formulation of it in terms of  $\mathcal{N} = 8$  superfields. After the original version of this paper appeared on the archives, two distinct proposals were made for an  $\mathcal{N} = 8$  superfield formulation: one an off-shell formulation of the abstract BLG theory [30, 31] using a ‘pure-spinor superspace’, the other an on-shell  $\mathcal{N} = 8$  superfield formulation of the Nambu bracket realization of the BLG theory [32]. Here we review the latter approach, with some simplifications, and we explain how the former approach extends to the Nambu-bracket

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<sup>1</sup>An  $\mathcal{N} = 1$  formulation of the abstract BLG theory was proposed previously in [29] but the Nambu bracket realization was not spelled out there.

realization of the BLG theory.

The low-energy dynamics of  $N$  coincident (or nearly-coincident) parallel planar D2-branes is an  $\mathcal{N} = 8$  supersymmetric  $D = 3$  gauge theory with gauge group  $SU(N)$ . As explained above,  $SU(N)$  can be viewed as a finite-dimensional approximation to  $SDiff_2$  (at least when  $M_2 = S^2$ ). It follows that the  $\mathcal{N} = 8$  supersymmetric Yang-Mills theory with gauge group  $SDiff_2$  may be interpreted as the field theory describing the low-energy dynamics of a D2-condensate, in much the same sense as the BLG theory describes an M2-condensate. In fact, we expect the renormalization group flow of a model for the D2-condensate to yield, in the infra-red limit, a model for the M2-condensate because this limit decompactifies IIA superstring theory to M-theory. Conversely, one might expect an  $S^1$ -compactification of a model for the M2-condensate to yield a model for the D2-condensate. Here we show that the  $\mathcal{N} = 8$  supersymmetric  $D = 3$   $SDiff_2$  Yang-Mills theory is indeed an  $S^1$ -compactification of the  $SDiff_3$  BLG theory, in a sense that we make precise. We also show that this model is an  $S^1$ -compactification, in a different sense, of the  $\mathcal{N} = 4$  supersymmetric  $D = 4$   $SDiff_2$  Yang-Mills theory mentioned above.

## 2. SDiff gauge theory

Let  $M_n$  be a closed  $n$ -dimensional real manifold that is compact with respect to some (non-dynamical) scalar density  $e$  in local coordinates  $\sigma^i$  ( $i = 1, \dots, n$ ). In new local coordinates  $\sigma^i + \xi^i(\sigma)$ , for infinitesimal vector field  $\xi$ , the scalar density becomes  $e - \partial_i(e\xi^i)$ , so the total volume is unchanged (as expected since this cannot depend on the choice of coordinate atlas for  $M_n$ ) but the local volume density changes unless we impose the constraint

$$\partial_i(e\xi^i) = 0. \tag{2.1}$$

The space of vector fields on  $M_n$  satisfying this constraint is a subalgebra of the algebra of all vector fields with respect to the Lie bracket of vector fields. It is the Lie algebra of the group  $SDiff(M_n)$  of ‘volume-preserving’ diffeomorphisms of  $M_n$ , which we abbreviate to  $SDiff_n$ .

We are concerned here with field theories in  $D$ -dimensional Minkowski spacetime, with cartesian coordinates  $x^\mu$ , and ‘mostly plus’ metric  $\eta_{\mu\nu}$ . Consider a scalar field  $\phi$  that is also a scalar on  $M_n$ ; it can be expanded in  $M_n$ -harmonics so  $\phi$  contains an infinity of Minkowski scalar fields, which transform among themselves under the infinite-dimensional group  $SDiff_n$ . The infinitesimal  $SDiff_n$  transformation of  $\phi$  is

$$\delta_\xi \phi = -\xi^i \partial_i \phi. \tag{2.2}$$

More generally, for any Minkowski-field  $T$  that is also a tensor on  $M_n$ , the infinitesimal  $SDiff_n$  transformation is

$$\delta_\xi T = -\mathcal{L}_\xi T, \tag{2.3}$$

where  $\mathcal{L}_\xi$  is the Lie derivative with respect to  $\xi$ . Besides (2.2), other important special cases are

$$\delta_\xi v^i = -\xi^j \partial_j v^i + v^j \partial_j \xi^i, \quad \delta_\xi \omega_i = -\xi^j \partial_j \omega_i - (\partial_i \xi^j) \omega_j. \tag{2.4}$$

for vector  $v^i$  and one-form  $\omega_i$  on  $M_n$ .

It is not difficult to construct Minkowski-space field theories that have a *rigid*  $\text{SDiff}_n$  invariance. For example, the Lagrangian density

$$\mathcal{L} = \int d^n \sigma e \left[ -\frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] \quad (2.5)$$

is  $\text{SDiff}_n$ -invariant, for any potential function  $V$ , as long as the  $M_n$ -vector parameter  $\xi$  is independent of the Minkowski space coordinates. This is an interacting Lagrangian density for the infinite number of Minkowski scalar fields contained in the  $M_n$ -harmonic expansion of  $\phi$ . However, we are interested in constructing  $\text{SDiff}_n$  *gauge* theories for which the  $\text{SDiff}_n$  invariance is local, in the sense that  $\xi$  is allowed to be an arbitrary Minkowski scalar in addition to being a divergence-free  $M_n$ -vector field. This will require new ingredients, as we explain next.

### 2.1 Local $\text{SDiff}_n$ invariance

The Minkowski spacetime derivative  $dT = dx^\mu \partial_\mu T$  is again a tensor on  $M_n$  (of the same type) as long as  $\xi$  is assumed to be independent of the Minkowski spacetime coordinates, but if we insist on *local*  $\text{SDiff}_n$  invariance then we need to use the covariant exterior derivative

$$\mathcal{D} = d + \mathcal{L}_s, \quad s = dx^\mu s_\mu{}^i \partial_i, \quad (2.6)$$

where the one-form-valued  $M_n$ -vector field  $s$  satisfies the constraint

$$\partial_i (e s^i) \equiv 0. \quad (2.7)$$

One may verify, for  $M_n$ -tensor  $T$ , that

$$\delta_\xi (\mathcal{D}T) = -\mathcal{L}_\xi (\mathcal{D}T), \quad (2.8)$$

provided that we assign to  $s$  the  $\text{SDiff}_n$  gauge transformation

$$\delta_\xi s = d\xi - [\xi, s], \quad (2.9)$$

where the bracket  $[,]$  indicates a commutator of vector fields on  $M_n$ . In particular, for  $M_n$ -scalar  $\phi$ ,

$$\mathcal{D}\phi = d\phi + s^i \partial_i \phi, \quad \delta_\xi (\mathcal{D}\phi) = -\xi^i \partial_i (\mathcal{D}\phi). \quad (2.10)$$

Note that the constraint (2.7) is  $\text{SDiff}_n$  invariant as a consequence of (2.1).

This formalism may be extended to tensor densities on  $M_n$ . In particular, the  $\text{SDiff}$  gauge transformation of the scalar density  $e$  is zero because of the constraint (2.1). We assume that  $e$  is independent of the Minkowski coordinates; i.e.

$$de = 0 \quad (\Leftrightarrow \partial_\mu e = 0). \quad (2.11)$$

As a consequence, one may show that<sup>2</sup>

$$\mathcal{D}(eT) = e\mathcal{D}T. \quad (2.12)$$

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<sup>2</sup>For example, both sides vanish when  $T = 1$ , the left hand side because  $de = 0$  and the right hand side by the definition of  $\mathcal{D}T$ .



As for Yang-Mills gauge theories, we may define the covariant 2-form field-strength of  $s$  as<sup>3</sup>

$$F = ds + \frac{1}{2} [s, s] . \quad (2.13)$$

This has the SDiff gauge transformation

$$\delta_\xi F = -[\xi, F] , \quad (2.14)$$

and it satisfies the ‘Bianchi’ identity  $\mathcal{D}F \equiv 0$ , i.e.

$$dF + [s, F] \equiv 0 . \quad (2.15)$$

We may write  $F = F^i \partial_i$ , where

$$F^i = ds^i + s^j \partial_j s^i . \quad (2.16)$$

This satisfies the additional identity

$$\partial_i (eF^i) \equiv 0 . \quad (2.17)$$

## 2.2 Pre-gauge invariance

The constraints (2.1) and (2.7) may be solved, locally, by writing

$$e\xi^i = \varepsilon^{ijk_1 \dots k_{n-2}} \partial_j \omega_{k_1 \dots k_{n-2}} , \quad es^i = \varepsilon^{ijk_1 \dots k_{n-2}} \partial_j A_{k_1 \dots k_{n-2}} , \quad (2.18)$$

where the  $(n-2)$ -form  $\omega$  (on  $M_n$ ) is an unconstrained parameter, and  $A$  is an  $(n-2)$ -form pre-potential on  $M_n$  (in addition to being a 1-form on the  $D$ -dimensional Minkowski spacetime); its SDiff $_n$  transformation is<sup>4</sup>

$$\delta_\xi A_{i_1 \dots i_{n-2}} = d\omega_{i_1 \dots i_{n-2}} - \xi^j \partial_j A_{i_1 \dots i_{n-2}} - (n-2) \partial_{[i_1} \xi^j A_{j|i_2 \dots i_{n-2}]} . \quad (2.19)$$

In addition, for  $n \geq 3$ , we have the abelian pre-gauge transformation<sup>5</sup>

$$A_{i_1 \dots i_{n-2}} \rightarrow A_{i_1 \dots i_{n-2}} + \partial_{[i_1} a_{i_2 \dots i_{n-2}]} \quad (2.20)$$

for a parameter  $a$  that is an  $(n-3)$ -form on  $M_n$ . The ‘pre-field-strength’ 2-form

$$G_{i_1 \dots i_{n-2}} = dA_{i_1 \dots i_{n-2}} + \frac{(n-1)}{2} s^j \partial_{[j} A_{i_1 \dots i_{n-2}]} \quad (2.21)$$

is SDiff $_n$  covariant and satisfies the ‘pre-Bianchi’ identity  $\mathcal{D}G \equiv 0$ . However, it is not pre-gauge invariant since

$$G_{i_1 \dots i_{n-2}} \rightarrow G_{i_1 \dots i_{n-2}} + d[\partial_{[i_1} a_{i_2 \dots i_{n-2}]}] . \quad (2.22)$$

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<sup>3</sup>We use the convention in which  $d$  acts ‘from the left’.

<sup>4</sup>In our convention, square brackets indicate antisymmetrization of the indices enclosed with ‘strength one’ (so that the brackets may be simply omitted on contraction of all antisymmetrized indices with some other antisymmetric tensor).

<sup>5</sup>This holds also for  $n = 2$  if we view the pre-gauge transformation as a shift of  $A$  by a closed  $(n-2)$ -form on  $M_n$ , in which case the  $M_2$ -scalar  $A$  is shifted by an arbitrary Minkowski 1-form that is constant on  $M_2$ .

The pre-gauge-invariant and SDiff covariant 2-form is the  $M_n$ -vector  $F^i$ , since

$$eF^i = \varepsilon^{ijk_1\dots k_{n-2}} \partial_j G_{k_1\dots k_{n-2}}. \quad (2.23)$$

We remark that the expression (2.21) is equivalent to

$$G_{i_1\dots i_{n-2}} = dA_{i_1\dots i_{n-2}} - \frac{1}{2(n-2)!} \epsilon_{jk i_1\dots i_{n-2}} s^j \wedge s^k, \quad (2.24)$$

where  $\epsilon_{i_1\dots i_n}$  are the components of an  $n$ -form  $\epsilon$  defined such that

$$\varepsilon^{i_1\dots i_n} \epsilon_{j_1\dots j_n} = e n! \delta_{j_1}^{[i_1} \dots \delta_{j_n}^{i_n]}. \quad (2.25)$$

### 2.3 Actions

Actions that are invariant under local SDiff $_n$  gauge transformations can be constructed from Minkowski space tensors that are also scalars on  $M_n$  via the SDiff covariant derivative. For example, the local SDiff $_n$  invariant version of (2.5) is

$$\mathcal{L} = \oint d^n \sigma e \left[ -\frac{1}{2} \eta^{\mu\nu} \mathcal{D}_\mu \phi \mathcal{D}_\nu \phi - V(\phi) \right]. \quad (2.26)$$

Given at least  $n$  scalar fields, potentials may also be introduced via the Nambu  $n$ -bracket: a possible SDiff $_n$  invariant potential for any  $n$  scalar fields  $(\phi_1, \dots, \phi_n)$  is

$$\mathcal{V} = \oint d^n \sigma e \{ \phi_1, \dots, \phi_n \}^2. \quad (2.27)$$

The main obstacle to the construction of SDiff $_n$  gauge theories is, for  $D > 1$ , the difficulty in finding a suitable ‘kinetic’ term for the SDiff $_n$  pre-gauge potential.<sup>6</sup> This difficulty appears insuperable for  $n \geq 4$ , so the main case of interest here will be  $n = 3$ . However, we begin with a review of the  $n = 2$  case.

#### 2.3.1 Gauge theories of area-preserving diffeomorphisms

For  $n = 2$ , the divergence-free constraint on the YM potential  $s$  implies, locally on  $M_2$ , that

$$e s^i = \varepsilon^{ij} \partial_i A, \quad (2.28)$$

where the scalar  $A$  is the pre-potential 1-form. Using this, we may rewrite the SDiff $_2$  covariant derivative as

$$\mathcal{D}\phi = d\phi - \{A, \phi\}, \quad (2.29)$$

where  $\{, \}$  is the Poisson bracket of functions on  $M_2$ ; i.e.

$$\{A, \phi\} := e^{-1} \varepsilon^{ij} \partial_i A \partial_j \phi. \quad (2.30)$$

We see that the SDiff $_2$  covariant derivative takes the form of a YM covariant derivative if we re-interpret  $A$  as a YM potential taking values in the infinite-dimensional Lie algebra of

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<sup>6</sup>For  $D = 1$  the pre-gauge potential is the Lagrange multiplier for the SDiff $_n$  constraints [27].

functions on  $M_2$  with respect to the Poisson bracket. This algebra is isomorphic to  $\text{SDiff}_2$  for  $M_2 = S^2$ ; for other topologies there is a finite number of divergence-free vector fields that cannot be written as in (2.28) but we ignore these here.

Now consider the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} \oint d^2\sigma \, e \eta^{\mu\rho} \eta^{\nu\sigma} G_{\mu\nu} G_{\rho\sigma}. \quad (2.31)$$

where  $G$  is the pre-field strength. Because of the isomorphism noted above, this is also a YM field-strength for  $A$ :

$$G = dA - \frac{1}{2} \{A, A\}, \quad (2.32)$$

The action is *not* invariant under the pre-gauge transformation  $G \rightarrow G + da$ , where  $a$  is a scalar on  $M_2$ , but this just means that the action includes a Maxwell action for a  $U(1)$  factor, which may be omitted because it is decoupled from the other fields.

### 2.3.2 Gauge theories of volume-preserving diffeomorphisms

For  $n = 3$  the  $\text{SDiff}$  pre-field-strength 2-form is

$$G_i = dA_i + s^j \partial_{[j} A_{i]}. \quad (2.33)$$

This is not a YM field strength. One might wonder, by analogy with the  $\text{SDiff}_2$  case, whether  $A_i$  takes values in *some* Lie algebra, presumably related to  $\text{SDiff}_3$ , but it appears that such a re-interpretation is not possible [33]. We are now dealing with an ‘exotic’ gauge theory. In view of this, it is not surprising that there is no longer any way to form a standard YM Lagrangian density. In any case,  $G_i$  is not pre-gauge invariant. However, the Minkowski scalar density

$$\mathcal{L} = \oint d^3\sigma \, e F_{\mu\nu}^i G_i^{\mu\nu} \quad (2.34)$$

is a possible kinetic term; it is both  $\text{SDiff}_3$  gauge invariant, manifestly, and pre-gauge-invariant as a consequence of the constraint (2.17). One may use this term to construct gauge theories that are analogous in many respects to standard Yang-Mills theories; in particular, one may construct simple supersymmetric gauge theories of volume-preserving diffeomorphisms in dimensions  $D = 3, 4, 6, 10$ .

Here we present the  $D = 10$  case for which the superpartner to the gauge prepotential  $A_i$  is a Majorana-Weyl spinor that is also a 1-form on  $M_3$ ; the result also applies, *mutatis mutandis*, for  $D = 3, 4, 6$ . Suppressing the Lorentz spinor index, we denote this superpartner by  $\chi_i$ , and we take  $\bar{\chi}_i$  to be the  $D = 10$  Majorana conjugate spinor. Let  $\Gamma^\mu$  be the  $D = 10$  Dirac matrices, and  $\Gamma^{\mu\nu}$  the antisymmetrized product of two of them (with ‘strength one’ convention for antisymmetrization). Now consider the SYM-like Lagrangian density

$$\mathcal{L} = \oint d^3\sigma \left[ e F_{\mu\nu}^i G_i^{\mu\nu} + i \epsilon^{ijkl} \partial_i \bar{\chi}_j \Gamma^\mu (\mathcal{D}_\mu \chi)_k \right]. \quad (2.35)$$

Using the pre-Bianchi identity  $\mathcal{D}G \equiv 0$ , and the usual  $D = 10$  Dirac-matrix identities, one can show that the corresponding action is invariant under the supersymmetry transformations

$$\delta A_{\mu i} = i \epsilon \Gamma_\mu \chi_i, \quad \delta \chi_i = e^{-1} \Gamma^{\mu\nu} \epsilon G_{\mu\nu i}, \quad (2.36)$$

where the parameter  $\epsilon$  is a constant anti-commuting Majorana-Weyl spinor.

This construction uses the fact that there is a natural bilinear inner product  $\langle | \rangle$  on the space of one-forms on  $M_3$ : the inner product of one-forms  $\omega$  and  $\omega'$  is

$$\langle \omega | \omega' \rangle := \oint d^3\sigma \epsilon^{ijk} \omega_i \partial_j \omega'_k. \tag{2.37}$$

However, this inner product is not positive semi-definite, and this means that the energy density will not be positive definite. A more physical class of  $\text{SDiff}_3$  gauge theories is possible for  $D = 3$ , as we explain in the following section.

### 2.3.3 $n \geq 4$

For  $n = 4$  the pre-field-strength  $G_{ij}$  is an abelian 2-form potential on  $M_4$ , and the field-strength  $F^i$  is (as always) a vector. As for  $n = 3$ , there is no way to construct an  $\text{SDiff}_4$  invariant from products of  $F^i$  alone, so  $G_{ij}$  must be used too but the possibilities are then severely restricted by the requirement of pre-gauge invariance. In fact, there are no  $\text{SDiff}_4$  and pre-gauge invariants that can be constructed from  $G_{ij}$  and  $F^i$  alone, and the same applies for  $n > 4$ . We will not pursue the possibility that such invariants exist once additional fields are introduced since we have not found anything useful in this way.

## 3. Conformal $\text{SDiff}_3$ gauge theories

There is an additional possibility for  $\text{SDiff}_3$  gauge theories that arises only for  $D = 3$ . Consider first, for a  $D = 4$  Minkowski spacetime, the Minkowski 4-form

$$L_{FG} = \oint d^3\sigma e F^i \wedge G_i. \tag{3.1}$$

This is manifestly  $\text{SDiff}_3$  gauge invariant, and pre-gauge invariant as a consequence of the constraint (2.17). One may show that, locally on Minkowski spacetime,

$$L_{FG} = dL_{CS} \tag{3.2}$$

where

$$L_{CS} = \oint d^3\sigma e \left[ ds^i \wedge A_i - \frac{1}{3} \epsilon_{ijk} s^i \wedge s^j \wedge s^k \right]. \tag{3.3}$$

Recall, as a special case of (2.25), that the alternating tensor  $\epsilon_{ijk}$  is defined by

$$\epsilon^{ijk} \epsilon_{lmn} = 6e \delta_\ell^i \delta_m^j \delta_n^k. \tag{3.4}$$

We note, for future use, that for any variation  $\delta A_i$  of  $A_i$ , one has

$$\delta L_{CS} = 2 \oint d^3\sigma e \delta A_i \wedge F^i - d \left[ \oint d^3\sigma e s^i \wedge \delta A_i \right]. \tag{3.5}$$

### 3.1 Chern-Simons-type gauge theories

We may now use  $L_{\text{CS}}$  as a Lagrangian 3-form for a  $D = 3$  Minkowski spacetime. This yields the Lagrangian density

$$\mathcal{L}_{\text{CS}} = \oint d^3\sigma e \epsilon^{\mu\nu\rho} \left[ (\partial_\mu s_\nu^i) A_{\rho i} - \frac{1}{3} \epsilon_{ijk} s_\mu^i s_\nu^j s_\rho^k \right]. \quad (3.6)$$

Omitting a total spacetime derivative, one has for arbitrary variation  $\delta A_{\mu i}$ ,

$$\delta L_{\text{CS}} = \oint d^3\sigma \left[ \epsilon^{\mu\nu\rho} \delta A_{\mu i} F_{\nu\rho}^i \right], \quad F_{\mu\nu}^i := 2 \left( \partial_{[\mu} s_{\nu]}^i + s_{[\mu}^j \partial_j s_{\nu]}^i \right). \quad (3.7)$$

One may use this result to verify that the action is both  $\text{SDiff}_3$  invariant and, because of the constraint (2.7), pre-gauge invariant; it is also conformal invariant if the  $M_3$  coordinates are inert and the pre-potential 1-form  $A_i$  is assigned conformal weight zero (as for the Minkowski-space exterior derivative  $d$ ). This action is analogous to the Chern-Simons (CS) term of a  $D = 3$  YM gauge theory, but the analogy is not complete because  $A_i$  is *not* a YM gauge potential, but rather its pre-potential, and for this reason we will say that it is of CS ‘type’. A peculiarity of this CS-type term is that it is parity-even rather than parity-odd because a parity flip in the  $D = 3$  spacetime can be compensated by a parity flip of  $M_3$ .

Suppose that we add to  $\mathcal{L}_{\text{CS}}$  the ‘matter’ Lagrangian density

$$\mathcal{L}_{\text{mat}} = \frac{1}{2} \oint d^3\sigma e (\mathcal{D}\phi)^2, \quad (3.8)$$

In this case, the variation of  $A_i$  yields the  $\text{SDiff}_3$ -invariant equation

$$\star F^i = -J^i \equiv -\frac{1}{2} e^{-1} \epsilon^{ijk} \partial_j \vec{\phi} \cdot D \partial_k \vec{\phi}. \quad (3.9)$$

Here we use the language of differential forms in  $D = 3$  Minkowski space with  $\star$  the Hodge dual operator.

### 3.2 $\text{SDiff}_3 \rightarrow \text{SDiff}_2$

Consider the following Lagrangian density

$$\mathcal{L} = \oint d^3\sigma e \left[ -\frac{1}{2} \eta^{\mu\nu} (\mathcal{D}_\mu \phi \mathcal{D}_\nu \phi) \right] + \frac{1}{2g} \mathcal{L}_{\text{CS}}, \quad (3.10)$$

where  $g$  is an arbitrary non-zero coupling constant. Let us suppose that the ‘internal’ 3-manifold of this theory takes the form

$$M_3 = M_2 \times S^1 \quad (3.11)$$

for some 2-manifold  $M_2$ . In this case we may split the local  $M_3$  coordinates such that

$$\sigma^i \rightarrow (\sigma^a, \sigma^*), \quad (a = 1, 2) \quad (3.12)$$

where  $\sigma^a$  are local coordinates for  $M_2$ , and  $\sigma^*$  is a local coordinate for  $S^1$ , periodically identified with unit period. We also have  $e = e_2 e_1$  where  $e_2$  is a scalar density on  $M_2$ , and we may choose  $e_1 = 1$  without loss of generality, so that  $e_2 = e$ .

If we suppose that  $\phi$  is periodically identified then

$$\phi \sim \phi + \sqrt{m} \tag{3.13}$$

for some mass parameter  $m$  since  $\phi^2$  has dimensions of mass in fundamental units. The  $\phi$  field now maps the  $S^1$  factor of  $M_3$  to another circle, so the  $\phi$  field space decomposes into a sum of spaces with distinct degree for this map. We will focus on the maps of degree one, for which

$$\phi = \sqrt{m} \sigma^* + \varphi, \tag{3.14}$$

where  $\varphi$  is a function on  $M_2$  only. The  $\text{SDiff}_3$  gauge variation of  $\varphi$  is

$$\delta_\xi \varphi = -\xi^a \partial_a \varphi - \sqrt{m} \xi^*. \tag{3.15}$$

This allows us to partially fix the  $\text{SDiff}_3$  gauge invariance by choosing

$$\varphi = 0 \quad (\Rightarrow \mathcal{D}\phi = \sqrt{m} s^*). \tag{3.16}$$

This restricts us to  $\text{SDiff}_3$  gauge transformations with  $\xi^* = 0$ ; i.e. the  $\xi^a$  transformations, but these are not yet those of  $\text{SDiff}(M_2)$  because  $\xi^a$  may still depend on  $\sigma^*$ . This is understandable because all fields may also still depend on  $\sigma^*$ .

To proceed, we will now dimensionally reduce by declaring that all fields (other than  $\phi$ ) are independent of  $\sigma^*$ . This is, of course, equivalent to keeping only the leading term in a Fourier expansion of all fields. In particular, we have  $\partial_* s^* = 0$ , so the constraint (2.7) reduces to

$$\partial_a (e s^a) = 0, \tag{3.17}$$

and hence  $s^*$  (actually its zero mode on  $S^1$ ) is now unconstrained. Moreover, the CS-type 3-form reduces to the sum of an exact 3-form and the 3-form

$$L_{\text{CS}} = 2 \oint d^2 \sigma e s^* \wedge G, \tag{3.18}$$

where  $G$  is the YM field strength 2-form:

$$G \equiv G_* = dA_* - \frac{1}{2} e^{-1} \varepsilon^{ab} \partial_a A_* \partial_b A_*. \tag{3.19}$$

Our starting Lagrangian density (3.10) now becomes

$$\mathcal{L} = -\frac{1}{2} \oint d^2 \sigma e \left[ m \eta^{\mu\nu} s_\mu^* s_\nu^* - \frac{1}{g} s_\mu^* \varepsilon^{\mu\nu\rho} G_{\nu\rho} \right]. \tag{3.20}$$

Eliminating  $s_\mu^*$ , we arrive at the Lagrangian density for an  $\text{SDiff}_2$  pure YM theory:

$$\mathcal{L} = -\frac{1}{4mg^2} \oint d^2 \sigma e G_{\mu\nu} G^{\mu\nu}. \tag{3.21}$$

As pointed out below (2.32), this is not pre-gauge invariant; as derived here, this follows from the fact that the dimensional reduction breaks pre-gauge invariance.

### 3.3 $\mathcal{N} = 1$ supersymmetry

It is straightforward to construct  $\mathcal{N} = 1$  supersymmetric actions invariant under  $\text{SDiff}_3$  gauge transformations for  $D = 3$ . We will need to introduce  $2 \times 2$  Dirac matrices  $\gamma^\mu$ , which we may choose such that

$$\gamma^{\mu\nu} = \varepsilon^{\mu\nu\rho} \gamma_\rho. \quad (3.22)$$

We will also need to introduce the  $D = 3$  charge conjugation matrix  $C$ , which is real antisymmetric, and equal to  $\gamma^0$  in a real representation for the Dirac matrices. Note that the matrices  $C\gamma^\mu$  are symmetric. For a Majorana spinor,  $\lambda$  say, the Dirac conjugate equals the Majorana conjugate, so

$$\bar{\lambda} = \lambda^t C, \quad (3.23)$$

where the superfix  $t$  indicates ‘transpose’.

Let us consider first the supersymmetric extension of the ‘CS’ term. This is

$$\mathcal{L}_{\text{CS}}^{\mathcal{N}=1} = \mathcal{L}_{\text{CS}} - \frac{i}{2} \oint d^3\sigma \varepsilon^{ijk} \bar{\chi}_i \partial_j \chi_k, \quad (3.24)$$

where  $\chi_i = dx^\mu \chi_{\mu i}$  is a Grassmann-odd 1-form on  $M_3$  that is also a  $D = 3$  Minkowski space Majorana spinor (we suppress spinor indices). The corresponding action is invariant under the infinitesimal supersymmetry transformations

$$\delta A_{\mu i} = \frac{i}{\sqrt{2}} \bar{\epsilon} \gamma_\mu \chi_i, \quad \delta \chi_i = -\frac{1}{\sqrt{2}} \gamma^{\mu\nu} \epsilon G_{\mu\nu i}, \quad (3.25)$$

where  $G_{\mu\nu i}$  are the components of the pre-field-strength 2-form  $G_i$ , and  $\epsilon$  is a constant anticommuting Majorana spinor parameter. The coefficient  $1/\sqrt{2}$  is introduced here for later convenience.

We may couple to this CS-type theory any number of scalar multiplets with component fields that are scalars on  $M_3$ . For simplicity, we consider a single scalar multiplet with scalar field  $\phi$  and two-component Majorana spinor field  $\psi$ . Consider the Lagrangian density

$$\mathcal{L}_0 = -\frac{1}{2} \oint d^3\sigma e \left[ \eta^{\mu\nu} \mathcal{D}_\mu \phi \mathcal{D}_\nu \phi + i \bar{\psi} \gamma^\mu \mathcal{D}_\mu \psi + (W')^2 - i W'' \bar{\psi} \psi \right], \quad (3.26)$$

for any real (superpotential) function  $W(\phi)$ . This is not supersymmetric by itself, but the Lagrangian density

$$\mathcal{L}_{\text{matter}}^{\mathcal{N}=1} = \mathcal{L}_0 - \frac{i}{\sqrt{2}} \oint d^3\sigma \varepsilon^{ijk} (\bar{\psi} \partial_j \chi_k) \partial_i \phi, \quad (3.27)$$

is invariant under the combined transformations of (3.25) and

$$\delta \phi = i \bar{\epsilon} \psi, \quad \delta \psi = (\gamma^\mu \mathcal{D}_\mu \phi + W') \epsilon. \quad (3.28)$$

If we now add these two  $\mathcal{N} = 1$  supersymmetric Lagrangian densities, introducing a coupling constant  $g$  to allow for different relative weights, we have

$$\mathcal{L} = \mathcal{L}_0 + \frac{1}{2g} \mathcal{L}_{\text{CS}} - i \oint d^3\sigma \varepsilon^{ijk} \left[ \frac{1}{4g} \bar{\chi}_i \partial_j \chi_k + \frac{1}{\sqrt{2}} \bar{\chi}_i \partial_j (\psi \partial_k \phi) \right]. \quad (3.29)$$

The  $\chi_i$  equation of motion determines  $\chi_i$  only up to a total  $M_3$  derivative because this is clearly a gauge invariance of the action; we may fix this gauge such that

$$\chi_i = -\sqrt{2}g\psi \partial_i\phi. \quad (3.30)$$

The net result is the Lagrangian density

$$\mathcal{L} = -\frac{1}{2} \oint d^3\sigma e \left[ \eta^{\mu\nu} \mathcal{D}_\mu\phi \mathcal{D}_\nu\phi + i\bar{\psi}\gamma^\mu \mathcal{D}_\mu\psi + (W')^2 - iW''\bar{\psi}\psi \right] + \frac{1}{2g} \mathcal{L}_{\text{CS}}. \quad (3.31)$$

This is invariant, omitting a total spacetime derivatives, under the infinitesimal supersymmetry transformations

$$\delta\phi = i\bar{\epsilon}\psi, \quad \delta\psi = (\gamma^\mu \mathcal{D}_\mu\phi + W')\epsilon, \quad \delta A_{\mu i} = -ig(\bar{\epsilon}\gamma_\mu\psi) \partial_i\phi. \quad (3.32)$$

The ‘CS’ term is essential for the invariance, and also needed is the Fierz identity

$$\gamma^\mu d\chi (d\bar{\chi}\gamma_\mu d\chi) \equiv 0. \quad (3.33)$$

### 3.4 Superspace

We now aim to recover the above model using superspace techniques. We begin by writing the superspace exterior derivative as

$$d = E^\alpha D_\alpha + E^\mu \partial_\mu \quad (3.34)$$

where  $(E^\alpha, E^\mu)$  are a basis of 1-forms on superspace such that the 2-component Majorana spinor derivative  $D_\alpha$  has the anti-commutator

$$[D_\alpha, D_\beta]_+ = 2i(C\gamma^\mu)_{\alpha\beta} \partial_\mu. \quad (3.35)$$

The fields  $(\phi, \chi)$  combine to form a single *superfield*  $\phi$  such that  $\sqrt{2}D_\alpha\phi = \chi_\alpha$ . As is customary, we use the same symbol to denote both a superfield and its first component since when these components are defined in terms of spinor derivatives (rather than by superfield expansion) each component equation may be interpreted as a superfield equation.

SDiff gauge fields are introduced via the SDiff covariant superspace exterior derivative

$$\mathcal{D} = d + \mathcal{L}_\Sigma, \quad \partial_i(e\Sigma^i) = 0, \quad (3.36)$$

where

$$\Sigma^i = E^\alpha \varsigma_\alpha^i + E^\mu s_\mu^i, \quad (3.37)$$

so that

$$\mathcal{D} = E^\alpha \mathbb{D}_\alpha + E^\mu \mathcal{D}_\mu, \quad (3.38)$$

where, for example,

$$\mathbb{D}_\alpha\phi = D_\alpha\phi + \varsigma_\alpha^i \partial_i\phi, \quad \mathcal{D}_\mu\phi = \partial_\mu\phi + s_\mu^i \partial_i\phi. \quad (3.39)$$



The components  $(\varsigma, s)$  of the superspace SDiff potential  $\Sigma$ , both of which are superfields, are related by the requirement that

$$[\mathbb{D}_\alpha, \mathbb{D}_\beta]_+ = 2i (C\gamma^\mu)_{\alpha\beta} \mathcal{D}_\mu, \quad (3.40)$$

which implies that

$$\mathbb{D}_{(\alpha\varsigma_\beta^i)} = D_{(\alpha\varsigma_\beta^i)} + \varsigma_{(\alpha}^j \partial_j \varsigma_{\beta)}^i = i (C\gamma^\mu)_{\alpha\beta} s_\mu^i. \quad (3.41)$$

Using this equation, one may show that the ‘matter’ Lagrangian density of (3.27) is reproduced, on elimination of auxiliary fields, by the *superspace* Lagrangian density

$$\mathcal{L}_{\text{matter}} = -\frac{1}{2} \oint d^3\sigma e [\bar{\mathbb{D}}\phi \mathbb{D}\phi - 4iW(\phi)]. \quad (3.42)$$

To verify this, one must use the superspace integration measure  $d^3x \frac{1}{8} [\bar{\mathbb{D}}, \mathbb{D}]$ .

To write the superspace Lagrangian for the CS-type term we first solve the divergence-free constraint on  $\Sigma$  by writing

$$\Sigma^i = e^{-1} \varepsilon^{ijk} \partial_j \Lambda_k, \quad (3.43)$$

where  $\Lambda_i$  is the superspace pre-potential; in terms of its (superfield) components  $(\lambda_i, A_i)$ , we have

$$\varsigma^i = e^{-1} \varepsilon^{ijk} \partial_j \lambda_k, \quad s^i = e^{-1} \varepsilon^{ijk} \partial_j A_k, \quad (3.44)$$

where

$$D_{(\alpha\lambda_\beta)_i} = i (C\gamma^\mu)_{\alpha\beta} A_{\mu i}. \quad (3.45)$$

Next, we introduce the *superspace* SDiff<sub>3</sub> field-strength 2-form

$$F^i = d\Sigma^i + \Sigma^j \partial_j \Sigma^i. \quad (3.46)$$

This can be written, locally on  $M_3$ , in terms of a *superspace* pre-field-strength  $G_i$  as

$$F^i = e^{-1} \varepsilon^{ijk} \partial_j G_k, \quad G_i = d\Lambda_i + \Sigma^j \partial_{[j} \Lambda_{i]}. \quad (3.47)$$

One may now show that (3.45) is equivalent to

$$G_{i\alpha\beta} = 0, \quad (3.48)$$

which is the pre-field strength analog of the standard Yang-Mills superspace constraints. This follows from

$$\begin{aligned} \Lambda_i &= E^\mu A_{\mu i} + E^\alpha \lambda_{\alpha i}, & D &= E^\mu D_\mu + E^\alpha D_\alpha, & \Sigma_i &= E^\mu A_{\mu i} + E^\alpha \lambda_{\alpha i} \\ G_i &= \frac{1}{2} E^\alpha \wedge E^\beta G_{\alpha\beta i} + E^\alpha \wedge E^\mu G_{\mu\alpha i} + \frac{1}{2} E^\mu \wedge E^\nu G_{\nu\mu i}, \end{aligned} \quad (3.49)$$

after taking into account that

$$dE^\alpha = -2i E^\alpha \wedge E^\beta (C\gamma^\mu)_{\alpha\beta}. \quad (3.50)$$

The *superspace* 4-form  $\oint d^3\sigma e F^i G_i$  is both SDiff<sub>3</sub> and pre-gauge invariant, but we cannot use it to construct directly the superspace integrand for the CS-type term. However,

using the techniques of [34, 35] we may map the ‘CS’ superspace 3-form to the CS-type Lagrangian density

$$\mathcal{L} = -4i \oint d^3\sigma e \left[ \bar{W}^i \lambda_i - \frac{1}{12} \epsilon_{ijk} \bar{\zeta}^i \gamma^\mu \zeta^j s_\mu{}^k \right], \quad (3.51)$$

where<sup>7</sup>

$$W^i = -\frac{i}{2} \gamma^\mu \mathcal{D}_\mu \zeta^i + \frac{i}{4} \bar{\mathbb{D}} \mathbb{D} \zeta^i. \quad (3.52)$$

One may verify that this reproduces (3.24) in the Wess-Zumino gauge.

#### 4. BLG

Let  $\phi^I$  ( $I = 1, \dots, 8$ ) be a Spin(8)  $\mathbf{8}_v$ -plet of real scalar fields, and  $\psi_A$  ( $A = 1, \dots, 8$ ) a Spin(8)  $\mathbf{8}_s$ -plet of Majorana anticommuting  $Sl(2; \mathbb{R})$  spinor fields, both on the cartesian product of 3-dimensional Minkowski spacetime with some 3-dimensional closed manifold without boundary,  $M_3$ . Let  $\rho^I$  be the  $8 \times 8$  Spin(8) ‘sigma’ matrices, and  $\tilde{\rho}^I$  their transposes, as in [21]. Note that

$$\rho^{IJ} := \rho^{[I} \tilde{\rho}^{J]} \quad (4.1)$$

is *antisymmetric* in its spinor indices. We also define

$$\tilde{\rho}^{IJK} := \tilde{\rho}^{[I} \rho^{JK]}, \quad \rho^{IJKL} := \rho^{[I} \tilde{\rho}^{JKL]}. \quad (4.2)$$

Now consider the following Lagrangian density

$$\begin{aligned} \mathcal{L}_{M2} = \oint d^3\sigma \left[ -\frac{1}{2} e |\mathcal{D}\phi|^2 - \frac{i}{2} e \bar{\psi} \gamma^\mu \mathcal{D}_\mu \psi + \frac{ig}{4} \epsilon^{ijk} \partial_i \phi^I \partial_j \phi^J (\partial_k \bar{\psi} \rho^{IJ} \psi) \right. \\ \left. - \frac{g^2}{12} e \{ \phi^I, \phi^J, \phi^K \}^2 \right] + \frac{1}{2g} \mathcal{L}_{CS}, \end{aligned} \quad (4.3)$$

where  $g$  is a real dimensionless parameter, and Spin(8) indices are suppressed. This Lagrangian density varies into a total spacetime derivative under the following infinitesimal supersymmetry transformations with  $\mathbf{8}_c$ -plet constant anticommuting spinor parameter  $\epsilon_{\dot{A}}$  ( $\dot{A} = 1, \dots, 8$ ):

$$\begin{aligned} \delta \phi^I &= i \bar{\epsilon} \tilde{\rho}^I \psi, & \delta A_{\mu i} &= -ig (\bar{\epsilon} \gamma_\mu \tilde{\rho}^I \psi) \partial_i \phi^I, \\ \delta \psi &= \left[ \gamma^\mu \rho^I \mathcal{D}_\mu \phi^I - \frac{g}{6} \{ \phi^I, \phi^J, \phi^K \} \rho^{IJK} \right] \epsilon. \end{aligned} \quad (4.4)$$

To verify this, one needs the ‘fundamental’ identity, and the Fierz identity

$$\tilde{\rho}^J \gamma^\mu d\psi (d\bar{\psi} \gamma_\mu d\psi) - \tilde{\rho}^I d\psi (d\bar{\psi} \rho^{IJ} d\psi) \equiv 0. \quad (4.5)$$

If all the fields of this model are expanded in harmonics on  $M_3$  then  $L$  becomes the sum of a Lagrangian  $L_0$  describing the centre of mass motion of the M2 condensate and a remainder that describes the ‘internal’ dynamics. The centre-of-mass fields come from the constant harmonic on  $M_3$ . There is no contribution of the constant harmonic to  $s^i$  since this is a vector on  $M_3$  (see e.g. [22]), so the centre of mass fields are those of a single  $\mathcal{N} = 8$  supermultiplet, with no interactions.

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<sup>7</sup>This quantity arises as the spinor field strength  $F_{\alpha\mu}{}^i = i(\gamma_\mu W^i)_\alpha$ ; it is the SDiff<sub>3</sub> counterpart of the spinorial SYM field strength, and it has its own pre-field strength  $\mathcal{W}_i$ , defined by  $W^i = e^{-1} \epsilon^{ijk} \partial_j \mathcal{W}_k$ .

#### 4.1 Fierz identity

Let us pause to prove (4.5). The l.h.s. can be rewritten by a Fierz rearrangement as

$$l.h.s. = \frac{1}{16} d\bar{\psi} \mathcal{O}^A d\psi [\tilde{\rho}^J \gamma^\mu \mathcal{O}^A \gamma_\mu - \tilde{\rho}^I \mathcal{O}^A \rho^{IJ}] d\psi, \quad (4.6)$$

where the overall sign is plus because  $d\psi$  is commuting, and  $\mathcal{O}^A$  is a complete set of the  $16 \times 16$  matrices formed by tensor products of  $(1, \gamma^\mu)$  with  $(1, \rho^{IJ}, \rho^{IJKL})$ . Actually, the only matrices of this type which contribute are those for which  $C\mathcal{O}^A$  is symmetric (because  $d\psi$  is commuting). This means that we have only to consider

$$\gamma^\mu \otimes 1, \quad 1 \otimes \rho^{IJ}, \quad \gamma^\mu \otimes \rho^{IJKL}. \quad (4.7)$$

It should be clear that the first two of these will produce terms of a type that already appear on the l.h.s. of (4.5) whereas the third does not. However, this ‘third’ matrix gives a contribution proportional to

$$d\bar{\psi} \gamma^\nu \rho^{KLMN} d\psi [-\tilde{\rho}^J \rho^{KLMN} - \tilde{\rho}^I \rho^{KLMN} \rho^{IJ}] \gamma_\nu d\psi \quad (4.8)$$

where we have used  $\gamma^\mu \gamma_\nu \gamma_\mu \equiv -\gamma_\nu$ . But this contribution is zero as a consequence of the identities

$$\rho^{IJ} \equiv \rho^I \tilde{\rho}^J - \delta^{IJ}, \quad \tilde{\rho}^I \rho^{KLMN} \rho^I \equiv 0. \quad (4.9)$$

This cancellation means that we now have

$$l.h.s. = \frac{1}{16} d\bar{\psi} \gamma^\nu d\psi [-\tilde{\rho}^J - \tilde{\rho}^I \rho^{IJ}] \gamma_\nu d\psi - \frac{1}{32} d\bar{\psi} \rho^{KL} d\psi [3\tilde{\rho}^J \rho^{KL} - \tilde{\rho}^I \rho^{KL} \rho^{IJ}] d\psi. \quad (4.10)$$

The overall minus sign of the second term arises because matrices like  $\rho^{12}$  square to *minus* the identity, and the additional factor of 1/2 compensates for the double counting implied by the index summation convention. Using the identities

$$\tilde{\rho}^I \rho^{IJ} \equiv 7\tilde{\rho}^J, \quad \tilde{\rho}^I \rho^{KL} \rho^{IJ} \equiv 4\rho^{KL} \tilde{\rho}^J - \tilde{\rho}^J \rho^{KL}, \quad [\tilde{\rho}^J, \tilde{\rho}^{KL}] = 4\delta^{J[K} \tilde{\rho}^{L]}, \quad (4.11)$$

we now find that

$$l.h.s. = -\frac{1}{2} (d\bar{\psi} \gamma^\nu d\psi) \tilde{\rho}^J \gamma_\nu d\psi + \frac{1}{2} (d\bar{\psi} \rho^{IJ} d\psi) \tilde{\rho}^I d\psi \equiv -\frac{1}{2} l.h.s., \quad (4.12)$$

from which it follows that  $l.h.s. = 0$ , which is just the Fierz identity (4.5).

Another way to prove the Fierz identity is to show that it follows from the  $D = 11$  Dirac-matrix identity that allows the construction of the  $D = 11$  supermembrane [36]. To see this, first write this  $D = 11$  identity in the form

$$\Gamma^{MN} d\Psi (d\bar{\Psi} \Gamma_M d\Psi) + \Gamma_M d\Psi (d\bar{\Psi} \Gamma^{MN} d\Psi) \equiv 0, \quad (4.13)$$

where  $\Psi$  is an anticommuting  $D=11$  Majorana spinors, and  $\Gamma^M$  are the  $D = 11$  Dirac matrices. Next, split the 11-vector index  $M \rightarrow (\mu, I)$ , breaking  $\text{Spin}(1, 10)$  to  $Sl(2; \mathbb{R}) \times \text{Spin}(8)$ , and consider the  $I$  component of the  $D=11$  identity for

$$\Psi = \begin{pmatrix} \psi \\ 0 \end{pmatrix}, \quad (4.14)$$

where the 16-component  $\psi$  transforms as the real  $(\mathbf{2}, \mathbf{8}_c)$  of  $Sl(2; \mathbb{R}) \times \text{Spin}(8)$ . This yields the identity (4.5).

## 4.2 Superconformal invariance

The Noether current corresponding to the invariance of  $\mathcal{L}_{M2}$  under the supersymmetry transformations (4.4) is

$$\mathcal{S}_{\text{Noether}}^\mu = \oint d^3\sigma \left[ \gamma^\nu \gamma^\mu \tilde{\rho}^I \psi \mathcal{D}_\nu \phi^I - \frac{g}{6} \gamma^\mu \tilde{\rho}^{IJK} \psi \{ \phi^I, \phi^J, \phi^K \} \right] \quad (4.15)$$

but we may add to this any vector spinor that is *identically* divergence-free. Consider, in particular, the ‘improved’ supersymmetry current

$$\mathcal{S}^\mu = \oint d^3\sigma \left[ \gamma^\nu \gamma^\mu \tilde{\rho}^I \psi \mathcal{D}_\nu \phi^I - \frac{g}{6} \gamma^\mu \tilde{\rho}^{IJK} \psi \{ \phi^I, \phi^J, \phi^K \} - \frac{1}{2} \tilde{\rho}^I \gamma^{\mu\nu} \partial_\nu (\phi^I \psi) \right], \quad (4.16)$$

which differs from the Noether current by the addition of the final term, which is identically divergence-free. As a consequence of this addition, one finds that the  $\psi$  equation of motion implies that

$$\gamma_\mu \mathcal{S}^\mu = 0. \quad (4.17)$$

This implies that  $\mathcal{S}^\mu$  is part of a supermultiplet that contains the ‘improved’, because trace-free, energy-momentum stress tensor, which in turn implies that the model is superconformal invariant.

Note that  $g$  cannot be set to zero in the action because of the CS term. In fact,  $|g|$  may be set to unity without loss of generality because, when  $|g| \neq 1$ , the scaling

$$A \rightarrow |g|^{2/3} A, \quad \sigma^i \rightarrow |g|^{1/3} \sigma \quad (4.18)$$

has the effect of taking  $|g| \rightarrow 1$ , except for an overall factor coming from the  $\oint d^3\sigma$  integral. The choice of sign of  $g$  is presumably related to whether we wish to describe a condensate of M2-branes or anti-M2-branes.

## 4.3 Equations of motion and the free-field limit

The equations of motion are

$$\begin{aligned} 0 &= \mathcal{D}^\mu \mathcal{D}_\mu \phi^I - i \frac{g}{2} e^{-1} \varepsilon^{ijk} \partial_i \phi^J \partial_j \bar{\psi} \rho^{IJ} \partial_k \psi + \frac{g^2}{2} \{ \{ \phi^I, \phi^J, \phi^K \}, \phi^J, \phi^K \}, \\ 0 &= \gamma^\mu \mathcal{D}_\mu \psi + \frac{g}{2} \rho^{IJ} \{ \phi^I, \phi^J, \psi \}, \\ 0 &= \frac{1}{2} \varepsilon^{\mu\nu\rho} F_{\nu\rho}^i + g e^{-1} \varepsilon^{ijk} \left[ \partial_j \phi^I \mathcal{D}^\mu \partial_k \phi^I - \frac{i}{2} \partial_j \psi \gamma^\mu \partial_k \psi \right]. \end{aligned} \quad (4.19)$$

Although we were unable to set  $g = 0$  in the action, this can be done in the equations of motion. The result is that  $F = 0$ , so that  $s$  is pure gauge. We may then choose a gauge for which  $s = 0$ , at which point we see that we have free field equations for  $\phi$  and  $\psi$ . These equations are those of a supersymmetric theory with transformations given by (4.4) for  $g = 0$  and  $s = 0$ .

#### 4.4 M2 boundaries

Bosonic configurations that preserve susy have a spinor  $\epsilon$  that obeys

$$\rho^I \gamma^\mu \mathcal{D}_\mu \phi^I \epsilon = \frac{g}{6} \{ \phi^J, \phi^K, \phi^L \} \rho^{JKL} \epsilon. \quad (4.20)$$

Let us choose  $M_3 = S^3$  and consider bosonic configurations for which

$$\phi^a = f(x^1) X_a(\sigma), \quad a = 1, 2, 3, 4 \quad (4.21)$$

where  $X_a$  are the functions that map  $M_3$  to the unit 3-sphere, as discussed in the action for general  $n$ : i.e.

$$\sum_{a=1}^4 X_a^2 = 1, \quad \{X_a, X_b, X_c\} = \epsilon^{abcd} X_d. \quad (4.22)$$

The field equation for the gauge potential  $A$  is then solved by  $s = 0$ , and the  $\phi$  equation reduces to

$$f'' = 3g^2 f^5. \quad (4.23)$$

This is solved by solutions of

$$f' = -gf^3, \quad (4.24)$$

which preserve 1/2 supersymmetry since the supersymmetry preservation condition (4.20) for such solutions reduces to

$$f (1 - \gamma^1 \rho_\star) \epsilon = 0 \quad (4.25)$$

where the matrix  $\rho_\star$ , defined by

$$\frac{1}{6} \epsilon^{abcd} \rho^{bcd} = \rho^a \rho_\star, \quad (4.26)$$

squares to the identity. Thus, we have 1/2 supersymmetric solutions<sup>8</sup> of the form

$$\phi^a = \frac{X_a(\sigma)}{\sqrt{2gx^1}} \quad (4.27)$$

with all other fields equal to zero [9].

Let  $T$  be the M2 tension, and define the rescaled field with dimensions of length,

$$\Phi^a = \phi^a / \sqrt{T}. \quad (4.28)$$

Because  $\sum_a (X_a)^2 = 1$ , we have

$$\sum_{a=1}^4 (\Phi^a)^2 = \left(1/\sqrt{2gTx^1}\right)^2, \quad (4.29)$$

which shows that at fixed  $x^1$  we have a 3-sphere of radius  $r = 1/\sqrt{2gTx^1}$ . This goes to infinity as  $x^1 \rightarrow 0$ , which means that the M2-branes have expanded to a planar 5-brane

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<sup>8</sup>Generic supersymmetric configurations have been classified in [38, 39].

at  $x^1 = 0$ . From the 5-brane perspective, there is a membrane ‘spike’ with 3-sphere cross section such that

$$x^1 = \frac{1}{2gT r^2}. \tag{4.30}$$

This solves the Laplace equation on  $\mathbb{E}^4$ , in polar coordinates  $(r, \theta, \varphi, \xi)$ . In other words we have a solution analogous to that found in [5] representing M2-branes ending on an M5-brane. The 5-brane tension was computed in [18] and shown to equal the M5-brane tension.

#### 4.5 D2 condensate from M2 condensate

Recalling that a D2-brane of IIA superstring theory is just an M2-brane of M-theory compactified on a circle [2], we should expect some analogous relation between the D2 and M2 condensates. The former is an  $N \rightarrow \infty$  limit of a maximally supersymmetric  $D = 3$  YM gauge theory with gauge group  $SU(N)$ ; as explained in the introduction, this limit yields an  $S\text{Diff}_2$  YM theory, so a D2-condensate is described (at low energy) by an  $\mathcal{N} = 8$  supersymmetric  $D = 3$  YM gauge theory with gauge group  $S\text{Diff}_2$ . We shall now exploit our earlier discussion of subsection 3.2 to show how this theory is obtained from the BLG  $S\text{Diff}_3$  gauge theory.

As in subsection 3.2, we choose  $M_3 = M_2 \times S^1$ , such that  $\sigma^a$  are local coordinates for  $M_2$  and  $\sigma^*$  is a coordinate for the  $S^1$  factor, periodically identified with unit period, and we take the density  $e$  to be a volume density for  $M_2$ . We then set

$$\phi^I = (\phi^{\mathcal{I}}, \phi^8) \quad (\mathcal{I} = 1, \dots, 7) \tag{4.31}$$

and periodically identify  $\phi^8$  with period  $\sqrt{m}$ . Again following subsection 3.2, we partially fix the  $S\text{Diff}$  gauge invariance by choosing

$$\phi^8 = \sqrt{m} \sigma^*, \tag{4.32}$$

and we then choose to consider only the zero modes on  $S^1$  of all other fields. Let us apply this generalized dimensional reduction<sup>9</sup> to the BLG theory. Relative to the discussion of subsection 3.2, there are several new ingredients. Firstly, there are an additional 7 scalar fields, for which

$$\mathcal{D}\phi^{\mathcal{I}} \rightarrow D\phi^{\mathcal{I}} := d\phi^{\mathcal{I}} - \{A, \phi^{\mathcal{I}}\}, \quad (A := A_*) \tag{4.33}$$

which is the YM covariant derivative for the group  $S\text{Diff}_2$ , realized via the Poisson bracket  $\{, \}$  of functions on  $M_2$ , as defined in (2.30). The  $S\text{Diff}_3$  covariant derivative of the spinor field  $\psi$  similarly reduces to an  $S\text{Diff}_2$  YM derivative. Secondly, there is a scalar potential

$$V := \frac{g^2}{12} \{\phi^I, \phi^J, \phi^K\}^2 \rightarrow \frac{mg^2}{4} \{\phi^{\mathcal{I}}, \phi^{\mathcal{J}}\}^2. \tag{4.34}$$

Finally there is the Yukawa-type term

$$i\frac{g}{4}\varepsilon^{ijk}\partial_i\phi^I\partial_j\phi^J(\partial_k\bar{\psi}\rho^{IJ}\psi) \rightarrow i\frac{g\sqrt{m}}{2}\varepsilon^{ab}\partial_a\phi^{\mathcal{I}}(\partial_b\bar{\psi}\rho^8\tilde{\rho}^{\mathcal{I}}\psi). \tag{4.35}$$

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<sup>9</sup>It is actually a supersymmetry-preserving variant of Scherk-Schwarz reduction similar to that considered in [37].

Here we have split the eight  $SO(8)$  sigma-matrices into  $\rho^8$  and the seven  $SO(7)$  sigma matrices  $\rho^{\mathcal{I}}$ . We thus find that

$$\mathcal{L}_{M2} \rightarrow \mathcal{L}_{D2} := \oint d^2\sigma e \left[ -\frac{1}{2} D^\mu \phi^{\mathcal{I}} D_\mu \phi^{\mathcal{I}} - \frac{1}{4mg^2} G_{\mu\nu} G^{\mu\nu} - \frac{mg^2}{4} \{ \phi^{\mathcal{I}}, \phi^{\mathcal{J}} \}^2 - \frac{i}{2} \bar{\psi} \gamma^\mu D_\mu \psi + i \frac{g\sqrt{m}}{2} \varepsilon^{ab} \partial_a \phi^{\mathcal{I}} (\partial_b \bar{\psi} \rho^8 \tilde{\rho}^{\mathcal{I}} \psi) \right]. \quad (4.36)$$

The corresponding action is invariant under transformations of  $\mathcal{N} = 8$  supersymmetry that may be deduced<sup>10</sup> from (4.4). As the  $SDiff_2$  gauge group may be viewed as an  $N \rightarrow \infty$  limit of  $SU(N)$ , it is natural to interpret  $\mathcal{L}_{D2}$  as the Lagrangian density describing the low-energy dynamics of a D2-condensate, related to the M2-condensate by reduction on the M-theory circle.

As a further check, we will now show that  $\mathcal{L}_{D2}$  is the dimensional reduction on  $T^7$  of a  $D = 10$  SYM theory with  $SDiff_2$  gauge group. The fields of the latter theory are a Minkowski 1-form potential  $A_m$  ( $m = 0, 1, \dots, 9$ ) and a Majorana-Weyl spinor  $\Psi$ , both scalars on  $M_2$ . Let  $\Gamma^m$  be  $D = 10$  Dirac matrices and  $\bar{\Psi}$  the  $D = 10$  Majorana-conjugate of  $\Psi$ . The  $D = 10$  Lagrangian density is

$$\mathcal{L}_{10} = \frac{1}{g_{10}^2} \oint d^2\sigma e \left[ -\frac{1}{4} G_{mn} G^{mn} - \frac{i}{2} \bar{\Psi} \Gamma^m D_m \Psi \right], \quad (4.37)$$

where  $g_{10}$  is a 10-dimensional coupling constant, and

$$G_{mn} = 2\partial_{[m} A_{n]} - \{A_m, A_n\}, \quad D_m \Psi = \partial_m \Psi - \{A_m, \Psi\}. \quad (4.38)$$

In fundamental units, the mass dimensions are

$$[A] = 1 \quad [\Psi] = \frac{3}{2}, \quad [g_{10}] = -3. \quad (4.39)$$

It may be verified that  $\mathcal{L}_{10}$  varies into a total spacetime derivative under the following infinitesimal supersymmetry transformations

$$\delta A_m = i\bar{\epsilon} \Gamma_m \Psi, \quad \delta \Psi = -\frac{1}{2} \Gamma^{mn} \epsilon G_{mn}. \quad (4.40)$$

To dimensionally reduce to  $D = 3$ , we choose real  $D = 10$  Dirac matrices of the form

$$\Gamma^\mu = \gamma^\mu \otimes \gamma^8, \quad \Gamma^{\mathcal{I}} = \mathbb{1}_2 \otimes \gamma^{\mathcal{I}}, \quad (4.41)$$

where  $(\gamma^{\mathcal{I}}, \gamma^8) = \gamma^I$  are the  $16 \times 16$   $SO(8)$  Dirac matrices, which we may write as

$$\gamma^I = \begin{pmatrix} 0 & \rho^I \\ \tilde{\rho}^I & 0 \end{pmatrix}. \quad (4.42)$$

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<sup>10</sup>In principle, it is necessary to include a compensating  $S^1$ -diffeomorphism to maintain the partial gauge choice (4.32), but this has no effect on the fields appearing in (4.36) as these are  $\sigma^*$ -independent  $M_2$ -scalars.

In this basis, the Majorana-Weyl spinor  $\Psi$  takes the form of (4.14). Dimensional reduction to  $D = 3$  of the Lagrangian density  $\mathcal{L}_{10}$  now yields  $\mathcal{L}_{D2}$  if we set

$$\frac{\text{Vol}(T^7)}{g_{10}^2} = \frac{1}{mg^2} \tag{4.43}$$

and

$$A_{\mathcal{I}} = \sqrt{m} g \phi^{\mathcal{I}}, \quad \Psi = \sqrt{m} g \psi. \tag{4.44}$$

Note that this implies that  $[\phi] = 1/2$  and  $[\psi] = 1$ , as expected for  $D = 3$  fields.

Naturally, if we compactify from  $D = 10$  on  $T^6$ , rather than  $T^7$ , we get a  $D = 4$   $\mathcal{N} = 4$  SDiff<sub>2</sub> gauge theory, and  $S^1$ -compactification of this theory yields the  $D = 3$   $\mathcal{N} = 8$  SDiff<sub>2</sub> gauge theory.

#### 4.6 $\mathcal{N} = 8$ superfields

Following the original version of this paper, an  $\mathcal{N} = 8$  superfield formulation of the Nambu bracket BLG field equations was found [32]; it consists of two coupled  $\mathcal{N} = 8$  superfield equations for the SDiff gauge field and the scalar superfield that is also a scalar on the three-dimensional manifold  $M_3$ . We shall now review this formulation.

We may define an SDiff<sub>3</sub>-covariant exterior derivative  $\mathcal{D}$  on  $\mathcal{N} = 8$  superspace exactly as for  $\mathcal{N} = 1$  superspace, by introducing the  $M_3$ -vector-valued 1-form potential  $\Sigma^i$ , which is now an  $\mathcal{N} = 8$  superfield: we now have the following decomposition generalizing (3.38):

$$\mathcal{D} = E^{\alpha\dot{A}} \mathbb{D}_{\alpha\dot{A}} + E^\mu \mathcal{D}_\mu, \tag{4.45}$$

where

$$\mathbb{D}_{\alpha\dot{A}} = D_{\alpha\dot{A}} + \varsigma_{\alpha\dot{A}}{}^i \partial_i, \quad \mathcal{D}_\mu = \partial_\mu + s_\mu{}^i \partial_i. \tag{4.46}$$

Here  $D_{\alpha\dot{A}}$  is the standard  $\mathcal{N} = 8$  superspace spinor derivative, and  $\varsigma_{\alpha\dot{A}}{}^i$  is the  $\mathbf{8}_c$ -plet of superpartners to the SDiff<sub>3</sub> gauge field  $s_\mu{}^i$ ; we shall confirm this below by showing that their respective field strengths are components of a field-strength superfield.

When acting on an  $M_3$ -scalar,

$$\mathcal{D}^2 = F^i \partial_i, \tag{4.47}$$

where  $F^i$  is the  $M_3$  vector-valued  $\mathcal{N} = 8$  field strength 2-form superfield. Equivalently, but in terms of the components of  $\mathcal{D}$  and  $F^i$ , we have

$$[\mathbb{D}_{\alpha\dot{A}}, \mathbb{D}_{\beta\dot{B}}]_+ = 2i \delta_{\dot{A}\dot{B}} (C\gamma^\mu)_{\alpha\beta} \mathcal{D}_\mu + F_{\alpha\dot{A} \beta\dot{B}}{}^i \partial_i \tag{4.48}$$

$$[\mathbb{D}_{\alpha\dot{A}}, \mathcal{D}_\mu] = F_{\alpha\dot{A} \mu}{}^i \partial_i \tag{4.49}$$

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] = F_{\mu\nu}{}^i \partial_i. \tag{4.50}$$

Following [32] we impose the constraint

$$F_{\alpha\dot{A} \beta\dot{B}}{}^i = 2i C_{\alpha\beta} W_{\dot{A}\dot{B}}{}^i, \tag{4.51}$$

where  $W_{\dot{A}\dot{B}}{}^i$  is in the  $\mathbf{28}$  of SO(8); it is also divergence-free, so

$$W_{\dot{A}\dot{B}}{}^i = -W_{\dot{B}\dot{A}}{}^i, \quad \partial_i (e W_{\dot{A}\dot{B}}{}^i) = 0. \tag{4.52}$$



Using the Jacobi identity

$$\left[ \mathbb{D}_{\alpha\dot{A}}, \left[ \mathbb{D}_{\beta\dot{B}}, \mathbb{D}_{\gamma\dot{C}} \right]_+ \right]_- + \left[ \mathbb{D}_{\beta\dot{B}}, \left[ \mathbb{D}_{\gamma\dot{C}}, \mathbb{D}_{\alpha\dot{A}} \right]_+ \right]_- + \left[ \mathbb{D}_{\gamma\dot{C}}, \left[ \mathbb{D}_{\alpha\dot{A}}, \mathbb{D}_{\beta\dot{B}} \right]_+ \right]_- \equiv 0, \quad (4.53)$$

one finds that

$$F_{\alpha\dot{A}\mu}{}^i = i(\gamma_\mu W_{\dot{A}}{}^i)_\alpha, \quad W_{\alpha\dot{B}}{}^i := \frac{i}{7} \mathbb{D}_{\alpha\dot{A}} W_{\dot{A}\dot{B}}{}^i, \quad (4.54)$$

and that

$$\mathbb{D}_{\alpha(\dot{A}} W_{\dot{B})\dot{C}}{}^i = i W_{\alpha\dot{D}}{}^i \left( \delta_{\dot{D}(\dot{A}} \delta_{\dot{B})\dot{C}} - \delta_{\dot{D}\dot{C}} \delta_{\dot{A}\dot{B}} \right). \quad (4.55)$$

Using the Jacobi identity

$$\left[ \mathcal{D}_\mu, \left[ \mathbb{D}_{\beta\dot{B}}, \mathbb{D}_{\gamma\dot{C}} \right]_+ \right]_- + \left[ \mathbb{D}_{\gamma\dot{C}}, \left[ \mathbb{D}_{\beta\dot{B}}, \mathcal{D}_\mu \right]_- \right]_+ + \left[ \mathbb{D}_{\beta\dot{B}}, \left[ \mathbb{D}_{\gamma\dot{C}}, \mathcal{D}_\mu \right]_- \right]_+ \equiv 0, \quad (4.56)$$

one finds that

$$F_{\mu\nu}{}^i = \frac{1}{8} \epsilon_{\mu\nu\rho} W^{\rho i}, \quad W_\mu{}^i := \frac{1}{2} \bar{\mathbb{D}}_{\dot{A}} \gamma_\mu W_{\dot{A}}{}^i, \quad (4.57)$$

and also that

$$\mathbb{D}_{\dot{A}(\alpha} W_{\beta)\dot{B}}{}^i = (C\gamma^\mu)_{\alpha\beta} (\mathcal{D}_\mu W_{\dot{A}\dot{B}}{}^i - 4\delta_{\dot{A}\dot{B}} W_\mu{}^i), \quad \mathbb{D}_{\alpha(\dot{A}} W_{\dot{B})}^\alpha{}^i = 0. \quad (4.58)$$

We see that the SDiff field strength supermultiplet includes a scalar **28** ( $W_{\dot{A}\dot{B}}{}^i$ ), a spinor **8<sub>c</sub>** ( $W_{\alpha\dot{A}}{}^i$ ) and a singlet divergence-free vector ( $W^{\mu i}$ ). There are many other independent components but these become dependent on-shell. The relevant Chern-Simons-like (CS-like) superfield equation *in the absence of ‘matter’ supermultiplets* is obviously  $W_{\dot{A}\dot{B}}{}^i = 0$ , since this sets to zero all SDiff<sub>3</sub> field strengths. We shall see below how this must be modified in the presence of ‘matter’.

We now introduce an **8<sub>v</sub>**-plet of scalar, and SDiff<sub>3</sub>-scalar, superfields  $\phi^I$ . The lowest component, which we also call  $\phi^I$ , may be identified with the BLG scalar fields. One then expects to find the superpartners in the next component, at least on-shell, and they should appear as the lowest component of an **8<sub>s</sub>**-plet of spinor superfields  $\psi_{\dot{A}}$ . We therefore impose the constraint<sup>11</sup>

$$\mathbb{D}_{\alpha\dot{A}} \phi^I = i \tilde{\rho}_{\dot{A}\dot{B}}^I \psi_{\alpha\dot{B}}. \quad (4.59)$$

Acting on this constraint with an SDiff<sub>3</sub>-covariant spinor derivative, and making use of the anticommutation relation (4.48), one finds that

$$\mathbb{D}_{\alpha[\dot{A}} \tilde{\rho}_{\dot{B}]C}^I \psi_C^\alpha = 2 W_{\dot{A}\dot{B}}{}^i \partial_i \phi^I, \quad (4.60)$$

which is solved by what was called in [32] the ‘super-CS’ equation

$$W_{\dot{A}\dot{B}}{}^i = \frac{2g}{e} \varepsilon^{ijk} \partial_i \phi^I \partial_j \phi^J \tilde{\rho}_{\dot{A}\dot{B}}^{IJ}. \quad (4.61)$$

It was shown in [32] that the two  $\mathcal{N} = 8$  superfield equations (4.59) and (4.61) imply the Nambu-bracket BLG equations (4.19).

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<sup>11</sup>This equation was called the *superembedding-like* equation in [32] because it can be obtained from the ‘superembedding’ equation for a single M2-brane [40] by first linearizing with respect to the dynamical fields in the static gauge, as in [41], and then covariantizing the result with respect to SDiff<sub>3</sub>.

## 5. Pure-spinor superspace

An *off-shell*  $\mathcal{N} = 8$  superfield formulation of the abstract BLG theory was proposed by Cederwall [30]. This formulation involves a ‘pure-spinor superspace’ for which there is an additional  $\mathbf{8}_c$ -plet<sup>12</sup> of *complex* commuting spinor coordinates  $\lambda_{\dot{A}}$  satisfying the ‘purity’ condition

$$\bar{\lambda}\gamma^\mu\lambda = 0, \quad (\bar{\lambda} := \lambda^t C) \tag{5.1}$$

where the *summed* Spin(8) indices have been suppressed. In other words, the pure-spinor superspace is parametrized by the standard  $\mathcal{N} = 8$   $D = 3$  superspace coordinates  $(x^\mu, \theta_{\dot{A}}^\alpha)$  together with  $\lambda_{\dot{A}}^\alpha$ . This is a variant of the  $D = 10$  pure-spinor superspace first proposed by Howe [42] and, from a more general perspective, a realization of the harmonic superspace programme of [43]. All pure-spinor superfields will be assumed to be *analytic* functions of  $\lambda$  that can be expanded as a Taylor series in powers of  $\lambda$ . Our aim here is to extend this formalism to the Nambu bracket realization of the BLG theory in which all pure-spinor superfields are additionally functions on the closed 3-manifold  $M_3$ .

### 5.1 Pure spinor Fierz identities

We begin by establishing some properties of the pure-spinor  $\lambda$ . The only analytic nonvanishing pure spinor bilinears are

$$M_{IJ} := \bar{\lambda}\tilde{\rho}^{IJ}\lambda, \quad N_{IJKL}^\mu := \bar{\lambda}\gamma^\mu\tilde{\rho}^{IJKL}\lambda. \tag{5.2}$$

For example,

$$\bar{\lambda}_{\dot{A}}\lambda_{\dot{B}} = \frac{1}{16}M^{IJ}\tilde{\rho}_{\dot{A}\dot{B}}^{IJ}, \quad \bar{\lambda}_{\dot{A}}\gamma_\mu\lambda_{\dot{B}} = \frac{1}{16 \cdot 4!}N_\mu^{IJKL}\tilde{\rho}_{\dot{A}\dot{B}}^{IJKL}. \tag{5.3}$$

It was stated in [30] that the constraint (5.1) implies the identity

$$M_{IJ}\rho^J\lambda \equiv 0. \tag{5.4}$$

This can be proved as follows. A Fierz transformation of the left hand side yields

$$M_{IJ}\rho^J\lambda = -\frac{1}{8}\rho^I\left(M_{PQ}\rho^{PQ}\lambda - \frac{1}{120}N_\mu^{PQRS}\gamma^\mu\rho^{PQRS}\lambda\right), \tag{5.5}$$

which implies that

$$M_{PQ}\rho_{PQ}\lambda = \frac{1}{120}N_{PQRS}^\mu\gamma_\mu\rho_{PQRS}\lambda. \tag{5.6}$$

A Fierz transformation of the left hand side of *this* equation leads, on using the identities

$$\rho_{PQ}\rho_{JK}\rho_{PQ} = -8\rho_{JK}, \quad \rho_{PQ}\rho_{JKLM}\rho_{PQ} = 8\rho_{JKLM}, \tag{5.7}$$

to the conclusion that

$$M_{PQ}\rho_{PQ}\lambda = \frac{1}{72}N_{PQRS}^\mu\gamma_\mu\rho_{PQRS}\lambda. \tag{5.8}$$

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<sup>12</sup>Actually,  $\mathbf{8}_s$  valued bosonic spinors were used in [30], but this is just a matter of convention.

Comparing (5.8) with (5.6), we see that

$$M_{PQ} (\rho_I \rho_{PQ}) \lambda = N_{PQRS}^\mu \gamma_\mu (\rho_I \rho_{PQRS}) \lambda = 0, \quad (5.9)$$

and using this in (5.5) we deduce (5.4).

The purity condition on  $\lambda$  also implies the following identities, the first of which was used in [30]:

$$(a) M_{[IJ} M_{KL]} = 0, \quad (b) N_{PQ[IJ} \cdot N_{KL]PQ} \equiv 0. \quad (5.10)$$

To prove these identities, it is convenient to begin by defining

$$M_{IJKLPQ} := \bar{\lambda} \tilde{\rho}^{IJKLPQ} \lambda = \frac{1}{2} \epsilon^{IJKLPQRS} M_{RS}, \quad (5.11)$$

and taking note of the following Spin(8) sigma-matrix identities

$$\begin{aligned} \bar{\lambda} (\tilde{\rho}_{[IJ} \tilde{\rho}^{PQ} \tilde{\rho}_{KL]}) \lambda &= M_{IJKLPQ} + 4M_{[IJ} \delta_K^P \delta_L^Q \\ \bar{\lambda} \gamma_\mu (\tilde{\rho}_{[IJ} \tilde{\rho}^{PQRS} \tilde{\rho}_{KL]}) \lambda &= 24N_\mu^{[RS} [IJ \delta_K^P \delta_L^Q], \end{aligned} \quad (5.12)$$

$$\begin{aligned} \bar{\lambda} (\tilde{\rho}_{IJKL} \tilde{\rho}^{PQ}) \lambda &= M_{IJKLPQ} - 12M_{[IJ} \delta_K^P \delta_L^Q \\ \bar{\lambda} \gamma^\mu (\tilde{\rho}_{IJKL} \tilde{\rho}^{PQRS}) \lambda &= -72N_\mu^{[RS} [IJ \delta_K^P \delta_L^Q], \end{aligned} \quad (5.13)$$

and

$$\begin{aligned} \bar{\lambda} (\tilde{\rho}_{IJKLMN} \tilde{\rho}^{PQ} \tilde{\rho}_{MN}) \lambda &= 4M_{IJKLPQ} + 144M_{[IJ} \delta_K^P \delta_L^Q \\ \bar{\lambda} \gamma_\mu (\tilde{\rho}_{IJKLMN} \tilde{\rho}^{PQRS} \tilde{\rho}_{MN}) \lambda &= -288N_\mu^{[RS} [IJ \delta_K^P \delta_L^Q]. \end{aligned} \quad (5.14)$$

Now, performing a Fierz transformation of the left hand side of (5.10b), we deduce, on using (5.12), that

$$M_{[IJ} M_{KL]} + \frac{1}{36} M_{RS} M_{IJKLRS} + \frac{1}{36} N_{PQ[IJ} \cdot N_{KL]PQ} = 0. \quad (5.15)$$

Next we note that the purity condition implies that

$$(\bar{\lambda} \gamma_\mu \lambda) N_{IJKL}^\mu = 0. \quad (5.16)$$

A Fierz transformation of the left hand side leads, on using (5.13), to the equation

$$M_{[IJ} M_{KL]} - \frac{1}{12} M_{RS} M_{IJKLRS} + \frac{1}{12} N_{PQ[IJ} \cdot N_{KL]PQ} = 0. \quad (5.17)$$

Finally, a Fierz transformation of  $M_{RS} M_{IJKLRS}$ , and use of (5.14), leads to the relation

$$M_{[IJ} M_{KL]} + \frac{1}{4} M_{RS} M_{IJKLRS} + \frac{1}{12} N_{PQ[IJ} \cdot N_{KL]PQ} = 0. \quad (5.18)$$

One can check that the system of three equations, (5.15), (5.17) and (5.18) for the three ‘variables’  $M_{[IJ} M_{KL]}$ ,  $M_{RS} M_{IJKLRS}$  and  $N_{PQ[IJ} \cdot N_{KL]PQ}$ , has only the trivial solution. This proves (5.10).

## 5.2 Off-shell BLG

Again following [30], we define the BRST-type operator

$$Q := \bar{\lambda}D, \tag{5.19}$$

which satisfies  $Q^2 \equiv 0$  as a consequence of the purity condition (5.1). We also introduce an  $M_3$ -vector-valued complex *anticommuting* scalar  $\Psi^i$ . In the present context,  $\Psi^i$  will play the role of the  $\text{SDiff}_3$  gauge potential; its  $\text{SDiff}_3$  gauge transformation, with commuting  $M_3$ -vector parameter  $\Xi$ , is

$$\delta\Psi^i = Q\Xi^i + \Psi^j\partial_j\Xi^i - \Xi^j\partial_j\Psi^i, \quad \partial_i(e\Xi^i) = 0. \tag{5.20}$$

We require that  $\partial_i(e\Psi^i) = 0$  so that, locally on  $M_3$ ,

$$\Psi^i = e^{-1}\epsilon^{ijk}\partial_j\Pi_k, \tag{5.21}$$

where  $\Pi_i$  is the complex *anticommuting*, and spacetime scalar, pre-gauge potential of this formalism. Note that, in contrast to the rather similar formalism of section 2, the gauge potential and pre-potential are Minkowski *scalars* (albeit anticommuting) rather than one-forms.<sup>13</sup>

Next, following our  $\mathcal{N} = 1$  superspace discussion at the end of subsection 3.4, we may introduce the field-strength superfield

$$\mathcal{F}^i := Q\Psi^i + \Psi^j\partial_j\Psi^i = e^{-1}\epsilon^{ijk}\partial_j\mathcal{G}_k, \tag{5.22}$$

where the last equality is valid locally on  $M_3$  and

$$\mathcal{G}_i := Q\Pi_i + \Psi^j\partial_j\Psi_i \tag{5.23}$$

is the pre-field-strength superfield of this formalism. Both  $\mathcal{F}^i$  and  $\mathcal{G}_i$  are  $\text{SDiff}_3$  covariant, so  $\mathcal{F}^i\mathcal{G}_i$  is an  $\text{SDiff}_3$  scalar and its integral is also pre-gauge invariant (i.e. invariant under  $\delta\Pi_i = \partial_i\alpha$  with an arbitrary anticommuting scalar  $\alpha$ ). Furthermore, this integral is  $Q$ -exact, in the sense that

$$\int d^3\sigma e \mathcal{F}^i\mathcal{G}_i = Q\mathbb{L}_{\text{CS}}, \tag{5.24}$$

where

$$\mathbb{L}_{\text{CS}} = \int d^3\sigma e \left( \Pi_i Q\Psi^i - \frac{1}{3}\epsilon_{ijk}\Psi^i\Psi^j\Psi^k \right) \tag{5.25}$$

is the CS-type Lagrangian density of this formalism; it is the Nambu-bracket version of the term proposed in [30] for the abstract BLG theory, although our construction is different. Note that  $\mathbb{L}_{\text{CS}}$  is both complex and anti-commuting.

We now introduce the  $\mathfrak{S}_v$ -plet of complex scalar  $\mathcal{N} = 8$  ‘matter’ superfields  $\Phi^I$ , with  $\text{SDiff}_3$  variation

$$\delta\Phi^I = \Xi^i\partial_i\Phi^I. \tag{5.26}$$

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<sup>13</sup>This is not so surprising when one recalls that the exterior product of ‘bosonic’ one-forms provides a representation of Grassmann algebra multiplication.

We allow these superfields to be complex because they may depend on the complex pure-spinor  $\lambda$  but, to make contact with the on-shell  $\mathcal{N} = 8$  superfield equations of subsection (4.6), we will need to impose a reality condition such that

$$\Phi^I = \phi^I + \mathcal{O}(\lambda) , \quad (5.27)$$

where  $\phi^I$  is a *real*  $\mathbf{8}_v$ -plet of ‘standard’  $\mathcal{N} = 8$  scalar superfields. We also define an SDiff<sub>3</sub>-covariant extension of  $Q\Phi^I$  by

$$\mathbb{Q}\Phi^I := Q\Phi^I + \Psi^i \partial_i \Phi^I . \quad (5.28)$$

We must use this SDiff<sub>3</sub>-covariant quantity to construct a ‘matter’ Lagrangian that can be added to the ‘CS’ term, which means that it must also be anti-commuting and analytic in  $\lambda$ . One possibility is

$$\mathbb{L}_{\text{mat}} = \frac{1}{2} M_{IJ} \oint d^3\sigma e\Phi^I \mathbb{Q}\Phi^J , \quad (5.29)$$

with  $M_{IJ}$  as defined in (5.2). To ensure manifest  $\mathcal{N} = 8$  supersymmetry one still needs to specify an adequate superspace integration measure. We refer to [31] for details of this measure, which has the crucial property of allowing us to discard a BRST-exact terms when varying with respect  $\Phi^I$ . This variation yields the superfield equation

$$M_{IJ} \mathbb{Q}\Phi^J = 0 , \quad (5.30)$$

which implies, as a consequence of the identity (5.10a), that

$$Q\Phi^I = \bar{\lambda} \tilde{\rho}^I \Theta \quad (5.31)$$

for some  $\mathbf{8}_s$ -plet of complex spinor superfields  $\Theta_{\alpha A}$ . The first nontrivial ( $\sim \lambda$ ) term in the  $\lambda$ -expansion of this equation is precisely the on-shell superspace constraint (4.59) with  $\psi = \Theta|_{\lambda=0}$ , which is real as a consequence of the assumed reality of  $\phi^I$ .

The combined SDiff<sub>3</sub>-invariant, complex and anti-commuting, Lagrangian density

$$\mathbb{L} = \mathbb{L}_{\text{mat}} - \frac{1}{g} \mathbb{L}_{\text{CS}} \quad (5.32)$$

is therefore a candidate for an off-shell  $\mathcal{N} = 8$  superfield formulation of the Nambu-bracket realization of the BLG theory, along the lines of [30]. The  $\Pi_i$  equation of motion of this combined Lagrangian is

$$\mathcal{F}^i = \frac{g}{2e} M_{IJ} \epsilon^{ijk} \partial_j \Phi^I \partial_k \Phi^J . \quad (5.33)$$

At this stage it is important to assume that  $\Psi^i$  has ‘ghost number one’ [30], which means that it is a power series in  $\lambda$  with vanishing zeroth order term (and similarly for its prepotential  $\Pi_i$ ). In other words

$$\Psi^i = \lambda_A^\alpha \zeta_{\alpha A}^i , \quad (5.34)$$

where  $\zeta^i$  is an  $M_3$ -vector-valued  $\mathbf{8}_c$ -plet of arbitrary anticommuting spinors. Its zeroth component in the  $\lambda$ -expansion is the fermionic SDiff<sub>3</sub> potential introduced, with the *same*

*symbol*, in (4.46). With this ‘ghost number’ assumption, (5.33) produces at lowest non-trivial order ( $\sim \lambda^2$ ) the superspace constraints (4.48) for the ‘ghost number zero’ contribution  $\zeta^i|_{\lambda=0}$  to the pure spinor superfield  $\zeta^i$  in (5.34), accompanied by the super CS equation (4.61) for the field strength  $W_{\dot{A}\dot{B}}$  constructed from this potential.

We have now shown how the on-shell  $\mathcal{N} = 8$  superfield formulation of subsection 4.6, and hence all BLG field equations, may be extracted from the equations of motion derived from the pure spinor superspace action (5.32). Of course, the field content and equations of motion should be analyzed at all higher-orders in the  $\lambda$ -expansion. Our results are consistent with the conjecture that the field equations of the action (5.32) are equivalent to those of the on shell superfield formulation of 4.6, in which case our results would imply that all higher-order fields in the  $\lambda$  expansion are auxiliary. Our results are also consistent with the weaker conjecture that all ‘higher-order’ fields are either auxiliary or decouple, in which case they might be removed by some ghost-number constraint. We shall not attempt to prove either of these conjectures here. Instead, we limit ourselves to the observation that a full analysis must take into account the existence of additional gauge invariances [30, 31]; in the present context, one may use the identities (5.10) to show that the BLG action is invariant under the infinitesimal transformations

$$\delta\Phi^I = \bar{\lambda}\tilde{\rho}^I\zeta_\alpha + (\mathbb{Q} + \Psi^j \partial_j)K^I, \quad \delta\Pi_i = K^I M_{IJ} \partial_i\Phi^J, \quad (5.35)$$

for arbitrary pure-spinor-superfield parameters  $\zeta_\alpha$  and  $K^I$ .

## 6. Discussion

It has been known for some time that there exist Yang-Mills gauge theories, in  $D$ -dimensional Minkowski spacetime, for which the gauge group is the infinite-dimensional group of area-preserving diffeomorphisms  $\text{SDiff}(M_2)$  of  $M_2$ , a closed two-dimensional manifold that is compact with respect to some volume form. The manifold  $M_2$  plays the role of an ‘internal’ space on which all Minkowski-space fields are also tensors, e.g. functions. Such models first arose for  $D = 1$  as gauge-mechanics models governing the light-cone-gauge dynamics of a relativistic membrane [23, 24]; it was later appreciated that the construction applies for any  $D$  [28]. A natural question is whether there exist gauge theories for which the gauge group is the group  $\text{SDiff}(M_n)$  of volume-preserving diffeomorphisms of some  $n$ -dimensional manifold  $M_n$  for  $n \geq 3$ ; we assume that  $M_n$  is closed and compact with respect to some volume  $n$ -form. Examples, with  $n = p$ , may be found for  $D = 1$  by light-cone gauge fixing of a relativistic p-brane [27], but no gauge-field kinetic term is required in this case. In this paper, we have developed a general formalism for the construction of  $D > 1$  gauge theories of  $n$ -volume-preserving diffeomorphisms. We ignored some global issues that distinguish between manifolds  $M_n$  of different topology, partly because we are mostly interested in the simplest case in which  $M_n$  is the  $n$ -sphere; for that reason we abbreviated  $\text{SDiff}(M_n)$  to  $\text{SDiff}_n$ .

The construction of a gauge-field kinetic term for an  $\text{SDiff}_n$  gauge theory is obstructed by the absence of a metric on  $M_n$  (as any metric could not be  $\text{SDiff}_n$  inert, it would have to be introduced as a dynamical variable and then we would have some GR-type theory

rather than a Minkowski field theory). As far as we can see, this obstacle is insuperable for  $n \geq 4$ , but there are options for  $n = 3$ . In particular, we have constructed a  $\text{SDiff}_3$  invariant analog of the  $D = 10$  super-YM theory. This theory is unphysical because the energy is not positive definite but it is nevertheless an example of an ‘exotic’  $D > 3$  Minkowski-space gauge theory; i.e. one not of YM type. This shows that the uniqueness of the YM minimal interaction for  $D > 3$  [44, 45] fails to apply when the number of massless vector fields is infinite. For  $D = 3$  there is another possibility for the construction of an  $\text{SDiff}_3$  invariant gauge-field kinetic term; this is an analog of the YM Chern-Simons (CS) term although the  $\text{SDiff}_3$  version is parity even because a parity flip in Minkowski spacetime can be ‘undone’ by a parity flip in the ‘internal’ 3-space. We have shown how to construct a general class of  $\mathcal{N} = 1$  supersymmetric  $\text{SDiff}_3$  gauge theories with this CS-type kinetic term, in components and using superspace methods.

Of particular interest is the special case of the superconformal  $D = 3$   $\text{SDiff}_3$  gauge theory with maximal  $\mathcal{N} = 8$  supersymmetry, because this is the Nambu-bracket realization of the BLG theory [9, 10], which can be viewed as describing a ‘condensate’ of coincident planar M2-branes; this realization was first considered by Bagger and Lambert [18], but the CS-type term appears first in [20]. We have presented here the full Lagrangian and supersymmetry transformation laws in a simple form. Following the original version of this paper, an  $\mathcal{N} = 8$  superspace formulation of the  $\text{SDiff}_3$  gauge theory was proposed by one of us [32], and we have reviewed this work, presenting some additional simplifications. This formalism makes the  $\mathcal{N} = 8$  supersymmetry manifest, although only at the level of the equations of motion. An alternative off-shell  $\mathcal{N} = 8$  superfield formalism of the abstract BLG theory was proposed around the same time by Cederwall [30, 31]; his formalism uses fields defined on a pure-spinor extension of  $\mathcal{N} = 8$  superspace. We have shown here how this pure-spinor superspace formalism can be fused with our  $\text{SDiff}_3$  formalism to give an off-shell action for the M2 condensate, although we did not attempt a full analysis of the field content.

The BLG theory was found by requiring that the Basu-Harvey equation [8], proposed to describe  $N$  M2-branes ending on an M5-brane, should arise as a condition for preservation of  $1/2$  supersymmetry. The original equation is solved by a tube-like configuration with a ‘fuzzy’ 3-sphere cross-section but this fuzzy 3-sphere becomes a smooth 3-sphere in the Nambu-bracket realization [18]. Here we have verified that this ‘smoothed’ Basu-Harvey equation is an equation for preservation of  $1/2$  supersymmetry in the context of the  $\text{SDiff}_3$  invariant theory for an M2 condensate. This could be viewed as further evidence of the connection between the BLG theory and the M5-brane [19, 20] although we believe this connection has not yet been properly understood; our current views on this topic can be found in [21].

In the special case that  $M_3 = M_2 \times S^1$ , we have shown that one may perform a dimensional reduction of the  $\text{SDiff}_3$  invariant BLG theory to arrive at an  $\text{SDiff}_2$ -invariant  $D = 3$  Yang-Mills gauge theory with maximal supersymmetry, which we interpreted as a model governing the low-energy dynamics of a D2-brane condensate of IIA superstring theory; recall that  $\text{SDiff}_2$  may be loosely viewed as the  $N \rightarrow \infty$  limit of  $\text{SU}(N)$ , and that the low-energy dynamics of a collection of  $N$  planar D2-branes is governed by a maximally

supersymmetric  $D = 3$   $SU(N)$  gauge theory. Results of [46] suggest that different ways of taking the large  $N$  limit of  $SU(N)$  lead to different topologies for  $M_2$ , and we imagine that something similar might apply to  $M_3$  in the case of the M2-brane condensate. This issue is connected to the important question that we passed over in the introduction: the nature of the low-energy dynamics of  $N$  coincident planar M2-branes for *finite*  $N$ .

It is tempting to suppose that an action describing the infra-red dynamics of  $N$  coincident M2-branes can be obtained by some discretization of the Nambu-bracket 3-algebra of functions on  $S^3$ , but this idea runs into the difficulty, mentioned in the introduction, that there is no suitable sequence of finite-dimensional metric Filippov 3-algebras labelled by  $N$ . There have been several proposals to circumvent this difficulty. One is to consider other types of algebra, e.g. [47]. Another is to allow non-metric Filippov 3-algebras, which means that one is restricted to consider equations of motion; in this scheme there is a natural explanation for the expected  $N^{3/2}$  scaling of the number of degrees of freedom with the number  $N$  of M2-branes [48] (see also [49]). Basically, fields on  $S^3$  become  $n \times n \times n$  ‘cubic matrices’ with  $\sim n^3$  degrees of freedom. However, the potential vanishes for fields on  $S^3$  that depend on only two of its coordinates, and these become ‘standard’  $n \times n$  matrices with  $\sim n^2$  degrees of freedom. The moduli space of vacua therefore has dimension  $\sim n^2$ , so that the number  $N$  of M2-branes described by the model scales with  $n$  like  $n^2$ ; the number of degrees of freedom therefore scales with  $N$  like  $N^{3/2}$ , exactly as predicted by AdS/CFT [50].

This ‘success’ of the Nambu-bracket approach may be contrasted with currently popular ‘ABJM’ proposal that involves an  $U(N) \times U(N)$  CS theory at level  $k = 1$ , with bi-fundamental matter [51]; this model has a manifest  $\mathcal{N} = 6$  supersymmetry but is conjectured to be  $\mathcal{N} = 8$  supersymmetric. It is a ‘conventional’ theory in the sense that its construction does not involve 3-algebras, but it is strongly coupled and so one cannot expect to read off the degrees of freedom from the Lagrangian. This is just as well since the conventional gauge theory structure would lead one to expect the number of degrees of freedom to scale like  $N^2$ , so one is led to conjecture that this is reduced to  $N^{3/2}$  by strong coupling effects. Although there is considerable support for this proposal, e.g. [52–54], it seems to us that it is more like a restatement of the problem (to one of strong coupling dynamics) than a solution to it.

If the ABJM proposal is correct, as seems likely, it should be possible to take the limit of large  $N$  to find the theory describing the M2-condensate, which could then be compared with the  $SDiff_3$  gauge theory presented in detail here. However, this would involve taking *two* limits simultaneously, strong coupling and large  $N$ . Double limits are notoriously tricky; they may not commute. It seems quite possible that one such limit could yield the  $\mathcal{N} = 8$  supersymmetric  $SDiff_3$  gauge theory, so there is no logical contradiction between the Nambu bracket approach advocated here and the conventional CS approach of ABJM.

Another outstanding problem is the nature of the  $D = 6$  conformal field theory governing the low energy dynamics of  $N$  coincident M5-branes. In light of what we now know about multiple coincident M2-branes, it seems likely that this problem will simplify in the  $N \rightarrow \infty$  limit. Given that a condensate of M2-branes may be viewed, in some sense, as an M5-brane, then is there a similar sense in which an M5 condensate could be viewed as a



yet higher-dimensional M-brane? Recalling that the recent advances in the M2 case were prompted by the Basu-Harvey proposal that the boundary of multiple M2-branes on an M5-brane might be understood in terms of fuzzy 3-spheres, it is natural to reconsider the implications of the recent demonstration [55] that an M5-brane can have a boundary on an M9-brane, which is a boundary of the 11-dimensional bulk spacetime of M-theory; in this context we should mention that higher-dimensional generalizations of the Basu-Harvey equation have been considered in [49, 56].

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